An analytic FV pressure gradient discretization for avoiding thermobaric instability

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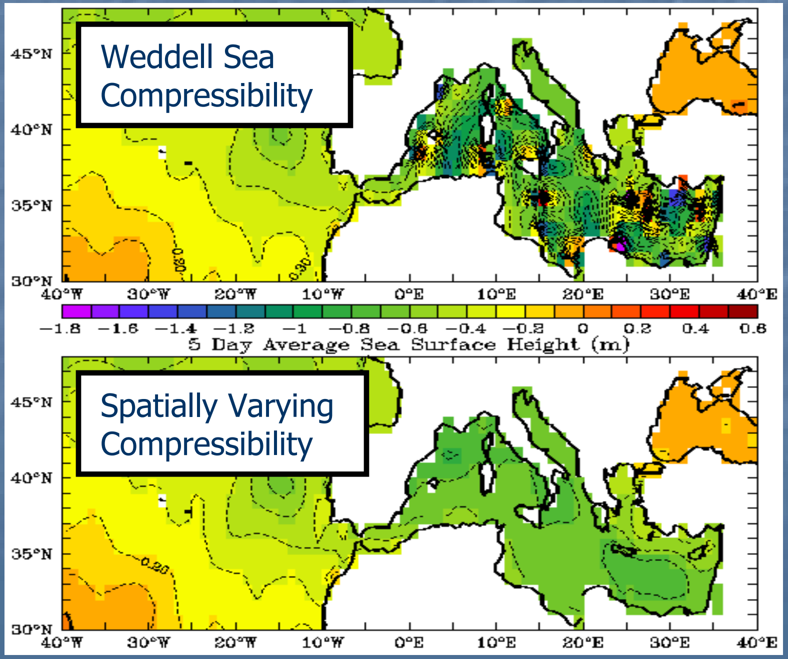
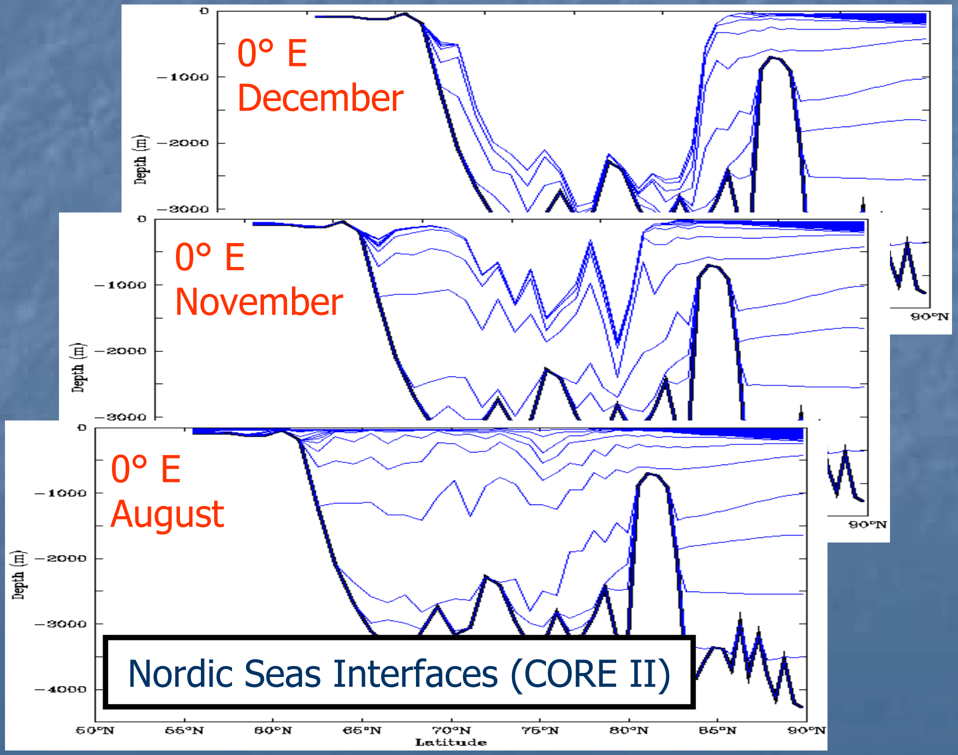
A (Previously Cured) Thermobaric Instability

- Compressibility is *awkward* for layer models (Sun et al., 1999; Hallberg 2005)
- Approximated EOS
  - → Wrong water masses
  - → Numerical problems

Thermobaric instability coupling with ML instability

- Only *after model drifts away from climatology*.
- Instability starts in interior interface heights
- Expels stratification to the surface where it is more easily destroyed
- Very young ages appear in both Northern and Southern high latitude abyss.





Bob's thought experiment

- Compressible E.O.S.
- Isothermal layers
  - Constant θ_1, θ_2
  - Discontinuous θ_k across interfaces

$k-\frac{1}{2}$ $\quad \Phi_{k-\frac{1}{2}}, p_{k-\frac{1}{2}}$

k $\quad \theta_k = \theta_1 \qquad \alpha_k = A(\theta_1, p) = A_1(p) \qquad \Phi_k(p) = \Phi_{k+\frac{1}{2}} - \int_{p_{k+\frac{1}{2}}}^{p} A_1(p')dp'$

$k+\frac{1}{2}$ $\quad \Phi_{k+\frac{1}{2}}, p_{k+\frac{1}{2}}$

$k+1$ $\quad \theta_{k+1} = \theta_2 \qquad \alpha_{k+1} = A(\theta_2, p) = A_2(p) \qquad \Phi_{k+1}(p) = \Phi_{k+\frac{1}{2}} - \int_{p_{k+\frac{1}{2}}}^{p} A_2(p')dp'$

$k+\frac{3}{2}$ $\quad \Phi_{k+\frac{3}{2}}, p_{k+\frac{3}{2}}$

$$\nabla_p \Phi_k(p) = \nabla_p \Phi_{k+\frac{1}{2}} - \int_{p_{k+\frac{1}{2}}}^{p} \underbrace{\nabla_p A_1(p')}_{=0} dp' - A_1(p)\underbrace{\nabla_p p}_{=0} + A_1\left(p_{k+\frac{1}{2}}\right)\nabla p_{k+\frac{1}{2}}$$

$$\nabla_p \Phi_{k+1}(p) = \nabla_p \Phi_{k+\frac{1}{2}} - \int_{p_{k+\frac{1}{2}}}^{p} \underbrace{\nabla_p A_2(p')}_{=0} dp' - A_2(p)\underbrace{\nabla_p p}_{=0} + A_2\left(p_{k+\frac{1}{2}}\right)\nabla p_{k+\frac{1}{2}}$$

Pressure gradient acceleration (PGA) anywhere within respective layer **independent** of position within layer

$$\boxed{\nabla_p \Phi_k - \nabla_p \Phi_{k+1} = \underbrace{\left[A_1\left(p_{k+\frac{1}{2}}\right) - A_2\left(p_{k+\frac{1}{2}}\right)\right]}_{N^2} \nabla p_{k+\frac{1}{2}}}$$

Acceleration difference between layers **should** be due only to gradients of middle interface $p_{k+1/2}$

Hallberg, OM '05

Bob’s thought experiment cont.

- The discrete version:

$$k-\tfrac{1}{2} \qquad \Phi_{k-\frac{1}{2}},\, p_{k-\frac{1}{2}}$$

$$\theta_k = \theta_1$$

$$k \qquad \alpha_k \approx \tilde{\alpha}_1 \equiv A_1(p_k) \qquad \Phi_{k-\frac{1}{2}} = \Phi_{k+\frac{1}{2}} + \Delta p_k \tilde{\alpha}_1(p_k)$$

$$k+\tfrac{1}{2} \qquad \Phi_{k+\frac{1}{2}},\, p_{k+\frac{1}{2}}$$

$$\theta_{k+1} = \theta_2$$

$$k+1 \qquad \alpha_{k+1} \approx \tilde{\alpha}_2 \equiv A_1(p_{k+1}) \qquad \Phi_{k+\frac{3}{2}} = \Phi_{k+\frac{1}{2}} - \Delta p_{k+1} \tilde{\alpha}_2(p_{k+1})$$

$$k+\tfrac{3}{2} \qquad \Phi_{k+\frac{3}{2}},\, p_{k+\frac{3}{2}}$$

$$\tilde{\alpha}_1 \approx \alpha_1\left(p_{k+\frac{1}{2}} - \tfrac{1}{2}\Delta p_k\right) \approx A_1\left(p_{k+\frac{1}{2}}\right) - \tfrac{1}{2}\Delta p_k \frac{\partial \alpha_1}{\partial p} + \ldots$$

$$\tilde{\alpha}_2 \approx \alpha_2\left(p_{k+\frac{1}{2}} + \tfrac{1}{2}\Delta p_{k+1}\right) \approx A_2\left(p_{k+\frac{1}{2}}\right) + \tfrac{1}{2}\Delta p_{k+1} \frac{\partial \alpha_2}{\partial p} + \ldots$$

$$PGA_k - PGA_{k+1} = (\tilde{\alpha}_1 - \tilde{\alpha}_2)\partial_x p_{k+\frac{1}{2}} + \tfrac{1}{2}\Delta p_k \partial_x \tilde{\alpha}_1 + \tfrac{1}{2}\Delta p_{k+1} \partial_x \tilde{\alpha}_2$$

$$\approx \underbrace{\left[A_1\left(p_{k+\frac{1}{2}}\right) - A_2\left(p_{k+\frac{1}{2}}\right)\right]}_{N^2}\partial_x p_{k+\frac{1}{2}} + \underbrace{\tfrac{1}{4}\left\{\Delta p_{k+1}\partial_x\left(\Delta p_{k+1}\frac{\partial \alpha_2}{\partial p}\right) - \Delta p_k \partial_x\left(\Delta p_k \frac{\partial \alpha_1}{\partial p}\right)\right\}}_{O(\Delta p)} + \ldots$$

Compressibility terms!

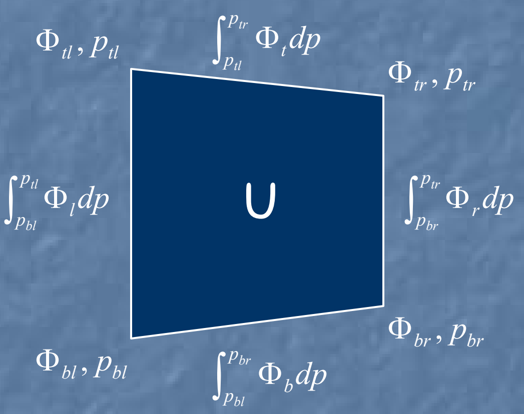
- These compressibility terms cause the acceleration to *spuriously* change when the interface is perturbed
  - thermobaric instability
  - inherent problem for “Lagrangian” coordinate models

Hallberg, OM ‘05

The atmospheric approach

- Atmospheric models are very compressible
  - Discretizations are similar
    - Finite volume

$$\iint \frac{\partial u}{\partial t}\, dxdp = -\iint \left.\frac{\partial \Phi}{\partial x}\right|_p dxdp + \ldots = -\int_1^2 \Phi dp - \int_2^3 \Phi dp - \int_3^4 \Phi dp - \int_4^1 \Phi dp + \ldots$$



    - 2nd order

$$\int_{p_X}^{p_Y} \Phi_l dp = \tfrac{1}{2}\left(\Phi_X + \Phi_Y\right)\left(p_Y - p_X\right)$$

$$\Phi_{tX} = \Phi_{bX} + \alpha_X\left(p_{bX} - p_{tX}\right)$$

  - Only difference is choice of variables

$$\frac{\partial u}{\partial t} = -\frac{-\iint \left.\frac{\partial \Phi}{\partial x}\right|_p dxd\pi}{\iint dxd\pi} + \ldots \qquad \pi = p^\kappa \qquad \delta_k \Phi = -c_p \theta_0 \delta_k p^\kappa$$

Linear in $\delta_k p^\kappa$

    - Use of the Exner function analytically incorporates E.O.S.
  - Recall: spurious compressibility terms vanish if linear E.O.S.
  - Ocean E.O.S. has no Exner analogue

e.g. Lin, QJRMS '95

Wright Equation of State

$$\boxed{\alpha = A + \frac{\lambda}{(P+p)}}$$

$$A = A(\theta, S)$$
$$P = P(\theta, S)$$
$$\lambda = \lambda(\theta, S)$$

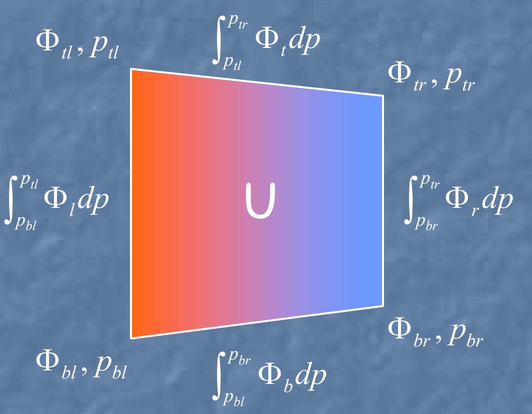
- Based on Tumlirz EOS
  - accurate and easy to integrate

  e.g. hydrostatic balance

$$\frac{\partial \Phi}{\partial p} = -\alpha$$

$$\Phi_{tX}(x) = \Phi_{bX}(x) + A(x)\left(p_{bX}(x) - p_{tX}(x)\right) - \lambda(x)\ln\left|\frac{P(x) + p_{tX}(x)}{P(x) + p_{bX}(x)}\right|$$

- Finite volume PGA

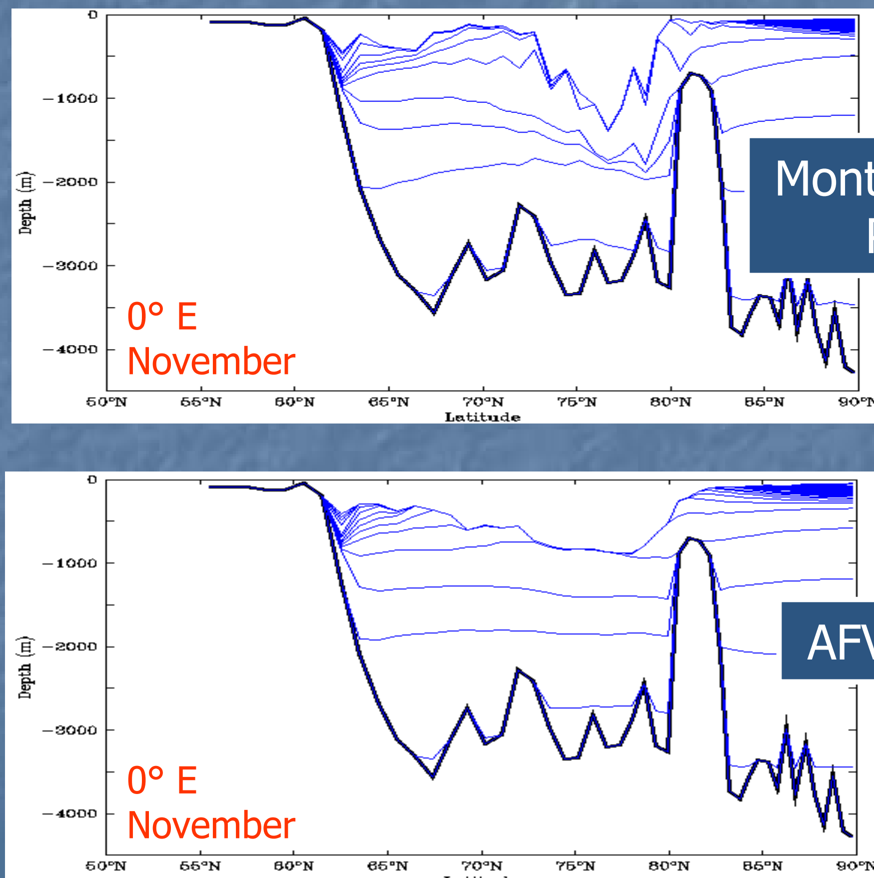


$$\int_{p_{bX}}^{p_{tX}} \Phi_X dp = \left(p_{tX} - p_{bX}\right)\left(\Phi_{bX} + \lambda_X - \tfrac{1}{2} A_X\left(p_{tX} - p_{bX}\right)\right) - \lambda_X\left(P_X + p_{tX}\right)\ln\left|\frac{P_X + p_{tX}}{P_X + p_{bX}}\right|$$

  - Assume piecewise constant θ and s within layer
  - Assume linear variation of θ and s in horizontal
  - Use sixth order Bode's rule for top/bottom integrals (tiny error w.r.t. analytic form)
  - Do series expansion of log and cancel leading terms

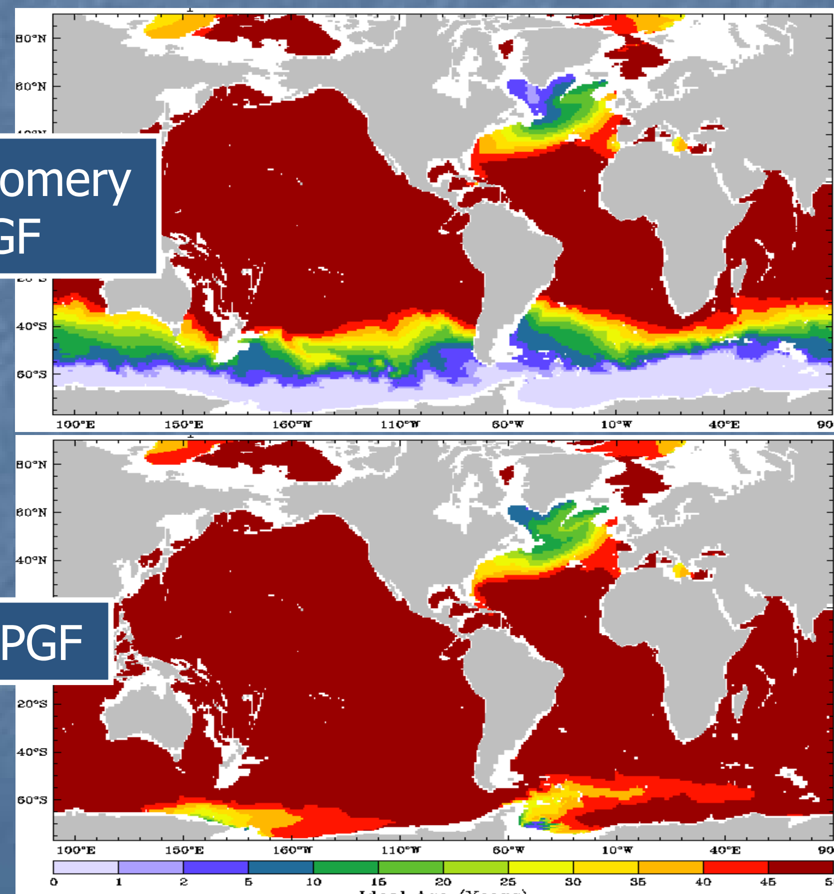
The results

- Nordic Sea thermobaric instabilities fixed
- Age distributions significantly improved



Montgomery PGF

AFV-PGF



Nordic Seas Interfaces (CORE II)

Ideal age at 2000m in CORE simulations (year 50)

Summary

- Found a simple approach to use "real" E.O.S.
  - (though mathematically challenging)

- Abandoned Montgomery potential in favour of FV method (Φ,p)
- No pressure gradient error for
  - Mass/pressure coordinates (p)
  - Geo-potential coordinates (z)
  - Isopycnal coordinates (ρ)
  - Or general coordinates in special cases

    e.g. Linear horizontal θ and s variation
  - PGE occurs only when assumptions used in analytic integration become inappropriate e.g. linear interpolation of θ, s

- Could/should extend to other E.O.S. forms
- Could use higher order representations of θ and s
  - within layer
  - in horizontal