Two-Grid Interpolation for the Los Alamos Hybrid Ocean Model: HYPOP

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Basic Idea behind HYPOP

Proposition (1): Natural vertical coordinate system for the momentum equation is geopotential (ie, Eulerian z-coordinates).

Proposition (2): Natural vertical coordinate system for continuity and tracer equations is Lagrangian (eg, Isopycnal).

Thus, solve momentum equation:

$$\rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + \nabla_3 \cdot \mathbf{u}_3\mathbf{u} + f\mathbf{k}\times\mathbf{u}\right) = -\nabla p^* + \mathbf{F}_u$$

on a z-grid, using interpolated pressure p^* from the Lagrangian grid. Solve continuity continuity and tracer equations on a Lagrangian grid using interpolated fluxing velocity $\mathbf{u}^*$ from the Eulerian grid:

$$\frac{\partial h}{\partial \tilde{t}} + \tilde{\nabla}\cdot\left(h\mathbf{u}^*\right) = 0, \quad \frac{\partial h\Theta}{\partial \tilde{t}} + \tilde{\nabla}\cdot\left(h\mathbf{u}^*\right)\Theta = 0, \quad \frac{\partial hS}{\partial \tilde{t}} + \tilde{\nabla}\cdot\left(h\mathbf{u}^*\right)S = 0$$

It is very desirable to interpolate p^* and $\mathbf{u}^*$ in an energy-consistent manner. That is, The flow of kinetic and potential energy should be consistent between the two grids.

(Note: The basis for the Eulerian momentum solution is POP, the Los Alamos Boussinesq, hydrostatic, B-grid model -- Hence, HYPOP.)

Alternative Approaches to an Energy-Consistent Interpolation

Eulerian Model

$$\frac{dKE}{dt} \leftrightarrow \underbrace{-\int \mathbf{u}\cdot\nabla_E p^* \, dV} \rightarrow \underbrace{\int p^* \nabla_E \cdot \mathbf{u} \, dV} \rightarrow \underbrace{\int \rho^* g w \, dV} \leftrightarrow \left(\frac{dPE}{dt}\right)$$

$$\frac{dKE}{dt} \leftrightarrow \overbrace{\left(-\int \mathbf{u}^* \cdot \nabla_L p \, dV\right)} \rightarrow \overbrace{\int p \nabla_L \cdot \mathbf{u}^* dV} \rightarrow \overbrace{\int \rho g w^* dV} \leftrightarrow \left(\frac{dPE}{dt}\right)$$

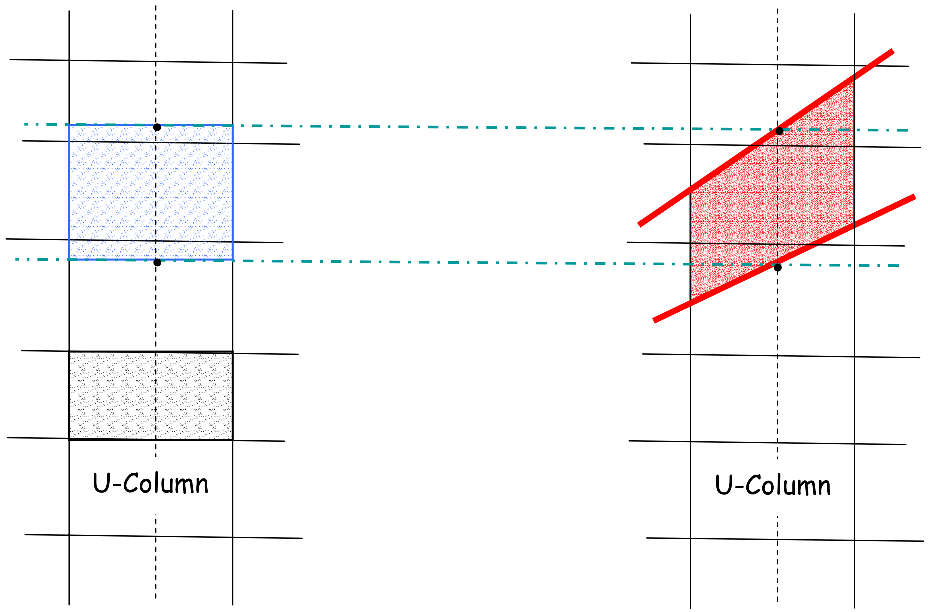
Lagrangian Model

PW PdV PEX

(Note: Boundary Terms Omitted)

Basic Idea

$$\int \mathbf{u}\cdot\nabla_E p^* dV \Rightarrow \int \mathbf{u}^* \cdot \nabla_L p dV, \quad \int \nabla_E p^* dV \Rightarrow \int \mathbf{u}^* \cdot \nabla_L p dV, \quad \left\langle \mathbf{u}^* \right\rangle = \int \mathbf{u}^* \cdot \nabla_L p dV \Big/ \int \nabla_L p dV$$



1. Eulerian grid Pressure Work and Pressure Gradient (ΔPW & ΔPG) in each cell are interpolated to the Lagrangian grid position for each cell (ΔPW & ΔPG)

2. Pressure Work and Pressure Gradient (ΔPW & ΔPG) at Lagrangian cell locations are taken to be equal to the Pressure Work and Pressure Gradient on the Lagrangian grid (ΔPW & ΔPG) since they represent the same quantity at the same location.

3. Lagrangian interpolated fluxing velocity is obtained for each cell as $u^* = \Delta PW / \Delta PG$

Outline of Procedure:
(A) Pressure Interpolation

$$\{p(z)\} \Rightarrow \{p^*(z)\}$$

Pressure, known at Lagrangian interfaces, is interpolated to Eulerian interfaces.

$$p^* \rightarrow \nabla_E p^*$$

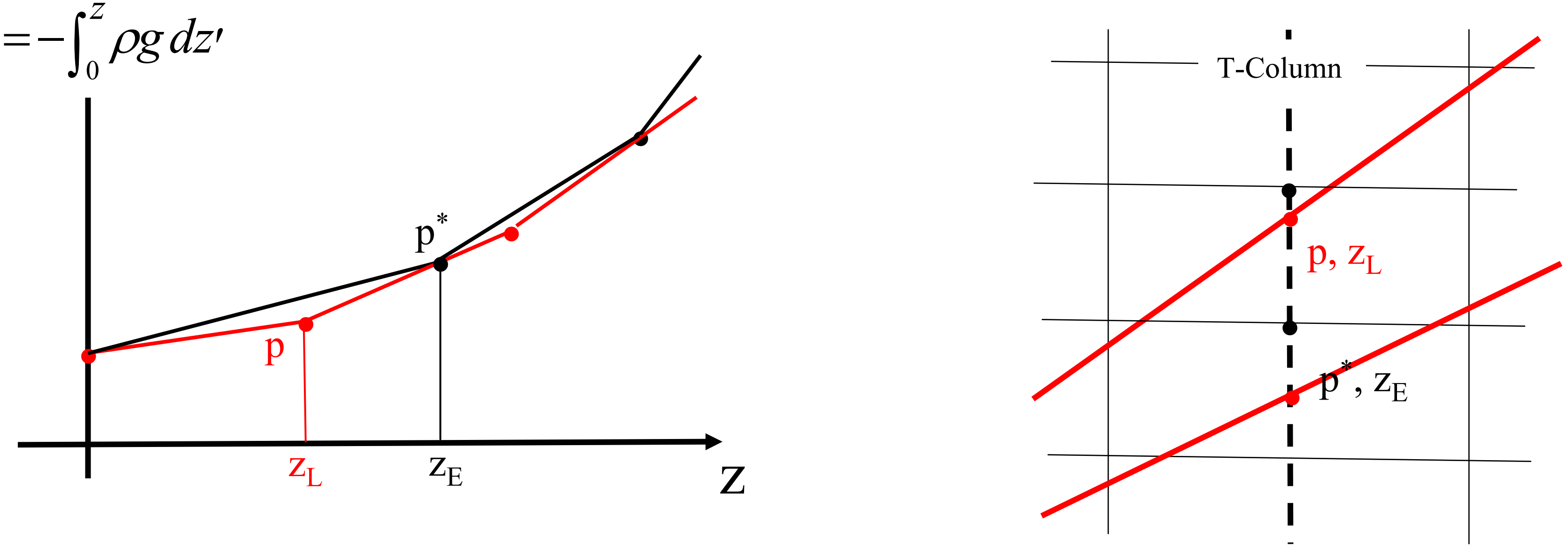
Knowing the Eulerian pressure, the Eulerian horizontal pressure gradient is computed. This is what is used in the momentum equation.

$$PW = \int \mathbf{u} \cdot \nabla_E p^* dz$$

Knowing the Eulerian horizontal pressure gradient, the pressure work in each column is calculated as a function of depth.

Interpolation of pressure in T-Column
(A 1D Remapping Interpolation)

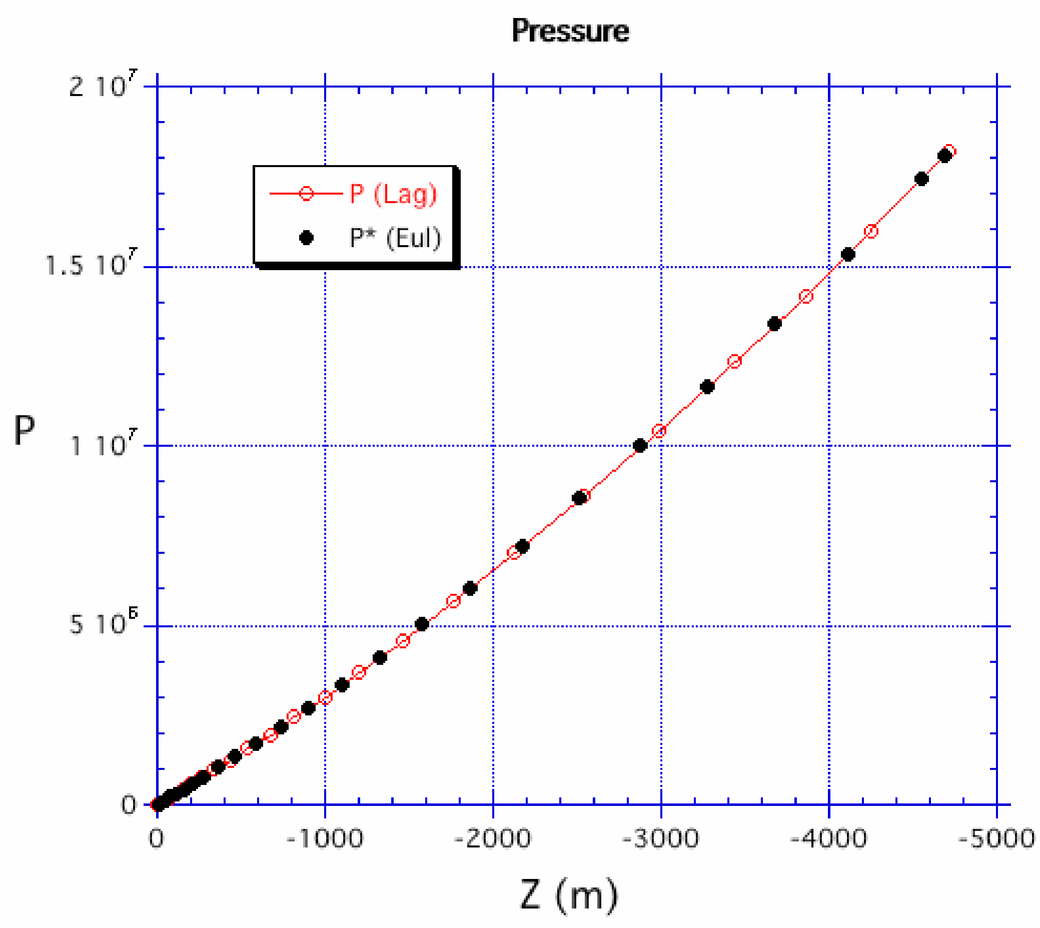
$$p = -\int_0^z \rho g \, dz'$$



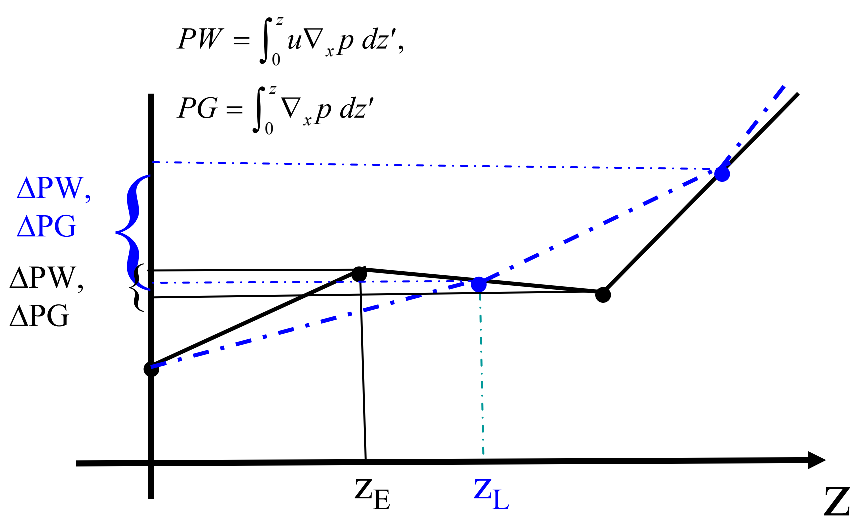
$$p \rightarrow p^*$$

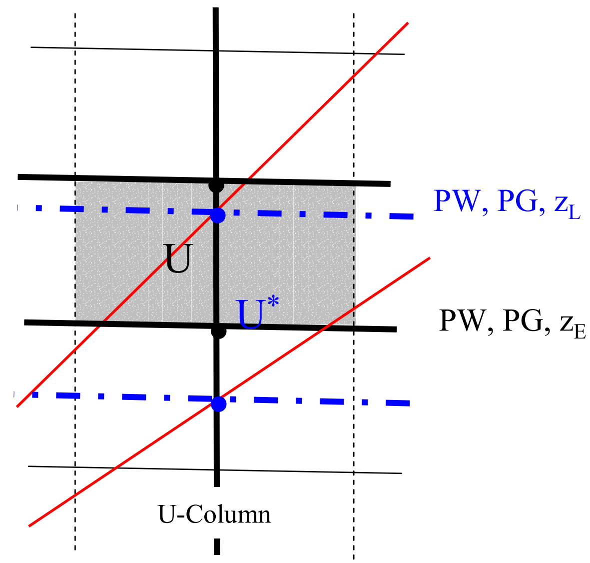
Note: Pressure interpolation preserves mass!

Pressure Interpolation



Interpolate Eulerian Pressure Work and Pressure Gradient to "Local/Horizontal" Lagrangian Grid (A 1D Remapping)





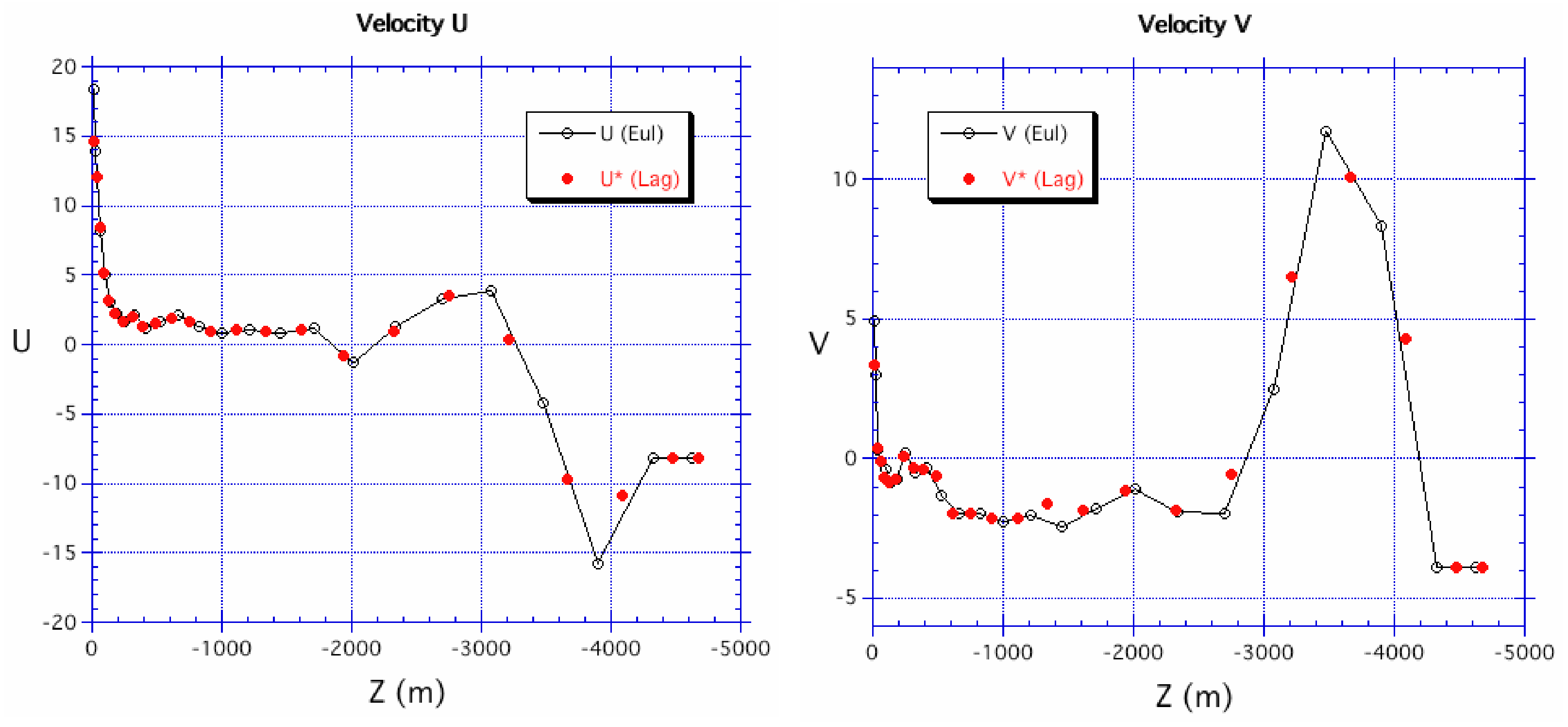
Note: This is the "correct" interpolated horizontal pressure work and pressure gradient (without σ-error) on the Lagrangian grid

Obtain Fluxing Velocity:

$$\left.\begin{aligned} \Delta PW &= \int_{\Delta z_L} \mathbf{u}^* \cdot \nabla_L p \, dz = \int_{\Delta z_L} \mathbf{u} \cdot \nabla_E p \, dz \\ \Delta PG &= \int_{\Delta z_L} \nabla_L p \, dz = \int_{\Delta z_L} \nabla_E p \, dz \end{aligned}\right\} \Rightarrow \langle \mathbf{u}^* \rangle = \frac{\int_{\Delta z_L} \mathbf{u}^* \cdot \nabla_L p \, dz}{\int_{\Delta z_L} \nabla_L p \, dz}$$

What is the corresponding Lagrangian continuity equation ?

Velocity Interpolation



Lagrangian continuity equation

Based on the discrete Lagrangian pressure gradient, the total Pressure Work on the entire grid is

$$PW = \sum A\overline{h}^{xy}\mathbf{u}^* \cdot \Delta_{\tilde{x}}\overline{p}^{ys} + g\sum \langle\rho\rangle A\overline{h}^{xy}\mathbf{u}^* \cdot \Delta_{\tilde{x}}\overline{z}^{ys}$$

Rearranging terms within the summation:

$$PW = -\underbrace{\sum A\,\overline{p}^{s}\Delta_{\tilde{x}}\cdot(\overline{\overline{h}^{xy}\mathbf{u}^*}^{y})}_{\text{Implies form of discrete Continuity Equation}} - \underbrace{g\sum A\,\overline{z}^{s}\Delta_{\tilde{x}}\cdot(\overline{\langle\rho\rangle\overline{h}^{xy}\mathbf{u}^*}^{y})}_{\text{Implies form of discrete rate of change of Potential Energy}}$$

This implies the following form for the conservation of volume and conservation of mass on the Lagrangian grid (not simultaneously valid):

$$\frac{\partial h}{\partial t} + \Delta_{\tilde{x}}\cdot(\overline{h}^{xy}\mathbf{u}^*) = 0 \quad (*), \qquad \frac{\partial \rho h}{\partial t} + \Delta_{\tilde{x}}\cdot(\overline{\langle\rho\rangle\overline{h}^{xy}\mathbf{u}^*}^{y}) = 0 \quad (**),$$

When mass is conserved (**), the total pressure work may be manipulated to give

$$PW = \underbrace{-\frac{d\,IE}{dt}}_{\text{Rate of change of Internal Energy}} + \underbrace{\sum A\,\overline{p}^{s}\frac{\partial h}{\partial t} + g\sum A\,\overline{z}^{s}\frac{\partial \rho h}{\partial t}}_{\text{Leads to rate of change of Potential Energy}}) \Rightarrow$$

Leads to conservation of Total Energy

In a Boussinesq model, (*) holds but not (**), and so PW does not exactly lead to energy conservation. However, (*) and (**) are consistent, and (*) becomes the appropriate " energy-consistent" Boussinesq Lagrangian continuity equation.

Is the Interpolated Fluxing Velocity Robust?

Unfortunately, the answer is "No"!

Recall that the velocity interpolation is

$$\langle \mathbf{u}^* \rangle = \int_{\Delta z_L} \mathbf{u}^* \cdot \nabla_L p \, dz \bigg/ \int_{\Delta z_L} \nabla_L p \, dz$$

This is an "interpolation" with a non-sign-definite weighting. It conserves Pressure Work and so is energy consistent, but it is ill-behaved when $\nabla_L p \to 0$ or changes sign.

Tests with simple models indicate that bursts of noise occur periodically. This is not acceptable behavior, and therefore this velocity interpolation is not acceptable.

There are two options:

(1) Retain energy consistency in the interpolation but change to a different approach: interpolate PdV or PEX. This may be feasible but much more difficult and complicated.

(2) Abandon energy consistency in favor of a simple, thickness-weighted interpolation. Is this a feasible solution?

Is Energy Consistent Interpolation Necessary?

We noted that an energy-consistent interpolation is highly desirable.
But is it necessary?

We note the following

- Originally (Bryan, 1969), the requirement for energy consistency arose from a need to avoid the non-linear (Phillips) instability.
- Energy exchange (involving the PW and PdV terms) occurs even in linear models of gravity waves. Strict energy consistency is usually not required in such models. The Lax Theorem states that consistency and stability (related to accuracy) are sufficient.
- Non-linear stability is ensured if advection terms do not contribute to the production of kinetic energy. This is easy enough to accomplish and it is completely independent of the "energy-consistency" requirement.

Thus, we conclude that energy consistency is probably not necessary for the interpolation between grids. Our preliminary tests with a simple model seem to confirm this.
(See next talk)

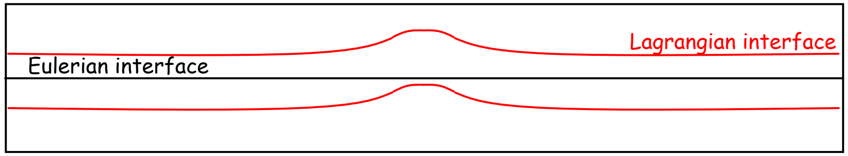
Some Properties of the Two-Grid Interpolation

An interesting property of the two-grid interpolation became apparent during testing:

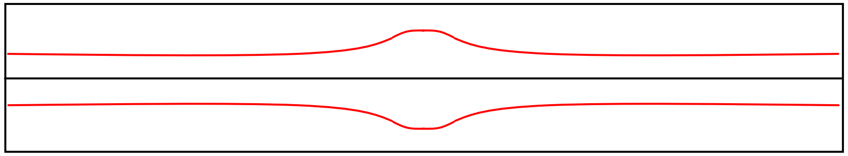
If the number of layers in the two grids is different, there can exist non-physical “null modes” that do not propagate!

Example:

Eulerian Grid	Lagrangian Grid
2 levels	3 layers
Supports 2 modes:	Supports 3 modes:
1 Barotropic	1 Barotropic
1 Baroclinic	2 Baroclinic (1 Propagating, 1 Non Propagating)



Propagating Lagrangian Mode



Non Propagating Lagrangian Mode

Since the Lagrangian grid has 3 degrees of freedom and the Eulerian grid only 2, there must be one mode on the Lagrangian grid that doesn’t couple to the Eulerian grid. This is the non-propagating mode or the null mode. The overall model supports only 2 physical modes.