A Physically Consistent Formulation of Lateral Friction in Shallow-Water Equation Ocean Models

ALEXANDER F. SCHHEPETKIN AND JAMES J. O'BRIEN

Centre for Ocean-Atmospheric Prediction Studies, * The Florida State University, Tallahassee, Florida

(Manuscript received 12 April 1995, in final form 20 November 1995)

ABSTRACT

Dissipation in numerical ocean models has two purposes: to simulate processes in which the friction is physically relevant and to prevent numerical instability by suppressing accumulation of energy in the smallest resolved scales. This study shows that even for the latter case the form of the friction term should be chosen in a physically consistent way. Violation of fundamental physical principles reduces the fidelity of the numerical solution, even if the friction is small. Several forms of the lateral friction, commonly used in numerical ocean models, are discussed in the context of shallow-water equations with nonuniform layer thickness. It is shown that in a numerical model tuned for the minimal dissipation, the improper form of the friction term creates finite artificial vorticity sources that do not vanish with increased resolution, even if the viscous coefficient is reduced consistently with resolution. An alternative numerical implementation of the no-slip boundary conditions for an arbitrary coast line is considered. It was found that the quality of the numerical solution may be considerably improved by discretization of the viscous stress tensor in such a way that the numerical boundary scheme approximates not only the stress tensor to a certain order of accuracy but also simulates the truncation error of the numerical scheme used in the interior of the domain. This ensures error cancellation during subsequent use of the elements of the tensor in the discrete version of the momentum equations, allowing for approximation of them without decrease in the order of accuracy near the boundary.

1. Introduction

The ocean exhibits a variety of different phenomena with extremely wide range spatial scales, which makes it impossible to resolve them all simultaneously in a numerical model. Consequently, small-scale processes are often parametrized as eddy viscosity, which has become a tuning parameter of numerical models rather than a physical concept. In many cases it is desirable to preserve conservation properties of the fluid motion in a numerical experiment. But even if one tries to simulate an inviscid fluid flow, dissipation is numerically inherent because of necessity to control leading-order dispersive truncation error, which causes nonphysical oscillations of the solution and eventually numerical instability (Hyman 1979; see also Leonard and Niknafs 1991; Haidvogel and Beckmann 1995, for a review). Does the particular form of viscous dissipation have any significance if one tries to make it perform this task with minimal distortion of the phenomenon of particular interest?

An example of a "purely technological" solution to the problem may be illustrated by the approach adopted in spectral simulations of two-dimensional turbulence. This is the scale-selective friction or hyperviscosity,

$$-\nu(-\nabla^2)^{\alpha}\mathbf{u},$$

where $\alpha = 1, 2, 3, \ldots$. The motivation is the following: in geophysical fluid motions, enstrophy and some fraction of energy cascades into small scales and is dissipated by molecular viscosity. In a computational model it necessary to remove energy from the smallest resolved scales in order prevent a nonphysical buildup. It was pointed out (Farge and Sadourny 1989) that in the context of shallow-water equations, the ratio of energy dissipation to enstrophy dissipation becomes smaller as the parameter $\alpha$ increases. This motivates the use of large values of $\alpha$ when one tries to reduce energy dissipation as much as possible. A typical value of $\alpha$ ranges to be from 2 to 8, while the nondimensional hyperviscosity coefficient $\nu = \nu(\mathcal{L}^{2(\alpha+1)}\right)^{-1}$ has a typical value of $10^{-18}$, $10^{-20}$. Here $\mathcal{L}$ and $\mathcal{L}$ are the velocity and length scales. It makes more sense to think about dissipation in terms of relative damping of the Nyquist wavenumber Fourier components per some characteristic timescale or in terms of grid-scale Reynolds number (Leonard 1991). One can construct an energy-conserving dissipative operator (Sadourny and Basdevant 1983; Arakawa and Hsu 1990), which dissipates enstrophy only.

© 1996 American Meteorological Society
Even though hyperviscosity effectively solves the numerical problem, it is of limited applicability. It has no clear physical background and when friction is physically relevant something else must be used instead of, or along with, the hyperviscous dissipation. Also, hyperviscosity brings higher-order derivatives into the system, and therefore it requires an additional nonphysical set of boundary conditions.

Historically, many of the numerical schemes adopted in ocean modeling originally appeared and were tested in atmospheric models (Lilly 1965; Grammeltvedt 1969; Sadourny 1975; Arakawa and Lamb 1981). Atmospheric models tend to parametrize subgrid-scale dissipation with a special design of the discrete schemes for the nonlinear terms rather than use of an explicit lateral dissipation (Arakawa and Hsu 1990). Consequently, atmospheric modelers did not pay sufficient attention to the question of explicit dissipation, although it was discussed theoretically (Obukhov 1962; Haynes and McIntyre 1987; Gent 1993; Schar and Smith 1993).

In contrast to atmospheric models, ocean models traditionally use a more or less realistic explicit version of dissipation. To some extent, this is explained by the inherent coupling of friction and boundary conditions in ocean models, which does not appear in atmospheric models. As a rule, the form of the viscous term is similar to molecular viscosity, except for the absolute value of the friction coefficient, which is often chosen for numerical reasons. In many cases, explicit lateral dissipation is implemented into a model based on property-conserving (e.g., mass, potential vorticity, energy, enstrophy) discrete numerical schemes. This implementation may be done without taking into account the fundamental physical principles, and, consequently, the conservation properties of the numerical model will be lost. In many cases simplified versions of the friction operator are used in order to decrease the computational cost (e.g., Killworth et al. 1991; Oberhuber 1993; Inoue and Welsh 1993), where some terms in the momentum equations were disregarded on the basis of scale analysis, which often destroys the symmetries of the original equations.

This motivates a more careful examination of the possible consequences of use of possible dissipation operators, both in continuous and discrete forms. From the physical point of view, one should distinguish several different phenomena associated with dissipation. Among them, horizontal viscous dissipation is the only one necessary in any ocean model for numerical reasons, and, therefore, it deserves special consideration in the context of the present paper.

2. The nature of dissipation

At first glance, the representation of horizontal friction in layered models appears trivial. In general, the shallow-water equations are derived by vertical integration of the primitive equations. The latter are obtained from the Navier–Stokes equations through scale analysis, neglecting the vertical acceleration versus the acceleration of gravity. Therefore, it should be possible to derive the friction term within the same procedure. However, this does not work (Gent 1993): the resulting form does not lead to a negatively definite term in the system’s kinetic energy budget. Moreover, we will soon see that this form of friction violates the momentum conservation principle. The derivation fails because the conventional hydrostatic approximation for the Navier–Stokes equations does not account for the difference between isopycnic and cross-isopycnic velocities versus horizontal and vertical velocities, as well as the fact that the emerging isopycnic system of coordinates is curvilinear.

Conservation properties of continuously stratified flows in the presence of dissipative forces and diffusion were considered theoretically (Obukhov 1962; see also Haynes and McIntyre 1987 for a review).1 Special attention was paid to potential vorticity conservation, which was generalized to the case of nonadvective (i.e., dissipative) fluxes. It was shown that there is no flux of potential vorticity through an isentropic surface even if there is mass flux across this surface (Haynes and McIntyre 1990). Due to dissipative fluxes, the isentropic surfaces do not correspond to the material surfaces in the general case. In the present paper, we shall restrict our study to the case when density interfaces are material surfaces. Therefore, all layer interfaces, the free surface, and the bottom surface are assumed to be impermeable for the dissipative fluxes. Diapycnal momentum flux will not be considered here, but it may be included into the model independently of lateral dissipation.

As an alternative to the direct derivation of the viscous term, one can formulate the fundamental properties of dissipation in the context of shallow water equations by applying the operation of vertical (strictly speaking, cross-isopycnic) integration to the constraints formulated in the more general theory of three-dimensional stratified flow, with the assumption that the viscous stress vanishes on the top and the bottom surface of the layer. Thus, regardless of the particular parametrization of the friction coefficient, the following are required.

---

1 Isopycinal diffusion of a passive scalar was discussed by Gent and McWilliams (1990) in the context of a three-dimensional density-coordinate system. In their approach the cross-isopycnic diffusive flux was assumed to be zero. It was shown that a standard approximation applied to the set of primitive equations, when diffusion in horizontal and vertical directions is treated independently, leads to a physically inconsistent interpretation of diffusion. Isopycinal and cross-isopycinal (diapycnal) diffusion terms should be introduced instead. This theory may be applied to the case of a layered system by formal substitution of $\nabla$ for $\nabla_{\Delta p}$.
(i) The momentum conservation principle requires that in the conservation (transport) form of the momentum equations, the lateral friction term must have the form of the divergence of a tensor.

(ii) Due to the angular momentum conservation law this tensor must be symmetric.

(iii) The friction must be dissipative. This implies that the dissipation in the right-hand side of the kinetic energy equation consists of two parts: term that has form of divergence of a vector and negatively defined source term.

(iv) The friction depends on the symmetric part of the local deformation tensor. Dependence of the antisymmetric part as well as the velocities or transports would mean that the viscous stress is not invariant under coordinate transformations, namely, transition to a uniformly rotating frame or to a frame moving with constant velocity.

To some extent, postulates (i)–(iv) are similar to the phenomenology of viscous stresses in compressible two-dimensional fluids (Landau and Lifshitz 1977; see also Tan Weiyan 1992), despite the fact that the nature of stratified flows is more complex.

As an immediate consequence from (i), one can see that the integral of the friction term over some two-dimensional domain can be converted to a boundary integral of the viscous momentum flux through the boundary. If the friction term cannot be represented in the divergence form, this means that there is a volume part of integral; that is, momentum is created by friction inside the domain, violating the momentum conservation principle. To emphasize the analogy between shallow-water equations and the general theory of stratified flows, we recall that the momentum flux in the shallow-water equations is already a vertically integrated quantity, so the two-dimensional integral may be interpreted as a volume integral over a three-dimensional domain bounded by two isopycnic surfaces.

If the stress tensor is not symmetric, as suggested in (ii), assuming some fluid element measured by \(dx \times dy\), as seen from above, it is possible to see that the torque produced by the viscous stress applied to the element is equal to the product of the difference of the off-diagonal elements of the tensor and the area of the element, \(dx \times dy\). According to the angular momentum conservation law, this torque is equal to the rate of change of the angular momentum of the fluid element, which is equal to the product of the moment of inertia of the element and the angular acceleration. As the moment of inertia is proportional to the area of the element squared, while the torque is proportional only to the area, the angular acceleration goes to infinity when the size of the fluid element goes to zero. In terms of fluid mechanics this would mean an infinitely large source of vorticity due to the friction. Consequences of the asymmetry of the viscous stress tensor in finite-difference models will be discussed later.

If (iii) is satisfied, the integral of the dissipation term over some domain may be expressed as a boundary integral and a negatively definite volume part, which means that energy is dissipated inside the domain.

As a consequence of (iv), the viscous stress identically vanishes for the velocity field like solid-body motion or solid-shell rotation of an infinitely thin layer on the surface of a sphere. It is interesting to note that the two-dimensional Laplace operator of a vector field on the surface of a sphere cannot be obtained from the three-dimensional operator by neglecting all terms that contain the radial component of the vector and the derivatives with respect the radius. This derivation would lead to a wrong geometrical term. Even though this term is usually small in geophysical applications, one can easily verify that as a consequence of this derivation, the solid-shell rotation would be subjected to uniform spin-down instead of being a stationary solution.

From the mathematical point of view, transition from the three-dimensional spherical coordinates to the two-dimensional geometry of the surface of a sphere is the transition from a Euclidian to a non-Euclidian geometry.

If (i) and (ii) are satisfied, it is possible to show that the dissipation term in the absolute vorticity equation has the form of the divergence of a vector (recall that vorticity is a scalar quantity in two dimensions). This means that absolute vorticity (which can be interpreted as content of potential vorticity in layer \(H\)) is not destroyed or created in the interior of the fluid by friction, but the net rate of change of absolute vorticity inside the domain is equal to the net flux of vorticity through the boundary. Thus, the potential vorticity and the absolute vorticity equations may be written as

\[
H \frac{D}{Dt} \left( \frac{\zeta + f}{H} \right) = \frac{D}{Dt} (\xi + f) + \frac{\partial u_a}{\partial x_a}
= \epsilon_{ab} \frac{\partial}{\partial x_a} \left( \frac{\partial \sigma_{bg}}{H} \frac{\partial \sigma_{bg}}{\partial x_b} \right) = \frac{\partial}{\partial x_a} \left[ \frac{1}{H} \frac{\partial}{\partial x_b} \left( \epsilon_{ab} \sigma_{bg} \right) \right]
\]  
(2)

In (2) and later in the paper lowercase letters \(u\) and \(v\), as well as \(u_a\), denote the components of the two-dimensional velocity vector, while uppercase \(U, V\) correspond to components of the mass flux (or transport). Term \(H\) is the layer thickness, so that \(U_a = H u_a\). Greek indices range from 1 to 2 and denote components of a vector or tensor. The summation rule over repeated indices is assumed. Here \(\zeta = \epsilon_{ab} \partial u_b / \partial x_a\) is the relative vorticity; \(f\) is the local Coriolis parameter; \(\sigma_{ab} = \sigma_{bg} \epsilon_{bg}\).

\footnote{Taking into account this analogy, we will use the words compressible and incompressible in the context of shallow waters meaning free-surface and rigid-lid approximation, respectively. The term incompressible shallow-water equations will be used exclusively in the context of the rigid-lid approximation of barotropic flow over topography.}
is the viscous stress tensor; $\epsilon_{ab}$ is the antisymmetric Levi–Civita pseudotensor, so that $\epsilon_{ab} = \beta - \alpha$ in Cartesian coordinates.

Consequences of (2) were studied by Schär and Smith (1993). It was shown that in addition to the usual (Laplacian) diffusion of vorticity, it yields two additional fluxes that depend on the mutual orientation of the gradients of layer thickness and vorticity as well as the gradients of layer thickness and velocity divergence. We should emphasize that the system of three-dimensional nonhydrostatic Navier–Stokes equations yields a similar property of the dissipation term in the vorticity equation, while the primitive equations do not retain this property. It is interesting to note that in three dimensions this issue requires special physical consideration (Obukhov 1962), but in the context of shallow water it is a formal consequence of the tensor form of the friction term.

3. Friction in numerical ocean models

The failure to obtain a physically consistent friction term from the three-dimensional primitive equations motivates development of a phenomenological approach to the dissipation term in shallow-water theory. This is similar to the classical formulation of the viscosity term in two- and three-dimensional hydrodynamics (Landau and Lifshitz 1977). In the context of shallow water theory this was discussed by Schär and Smith (1993). Following this idea, one may express the dissipation term in the momentum equations as divergence of viscous stress tensor,

$$
\sigma_{a\beta} = A_1 H \left( \frac{\partial u_a}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_a} - \delta_{a\beta} \frac{\partial u_\mu}{\partial x_\mu} \right) + A_2 H_{a\beta} \frac{\partial u_\mu}{\partial x_\mu}
$$

(3)

and prove that it has the properties (i)–(iv) postulated above.

Three other versions of the friction term are frequently used in ocean models (Hurlburt and Thompson 1980; Bleck and Boudra 1981; Killworth et al. 1991; Oberhuber 1993):

- $A H \Delta u$ (4)
- $\nabla (A H \nabla u)$ (5)
- $A \Delta U$ (6)

where $\Delta$ is the two-dimensional Laplace operator of a two-dimensional vector. Viscous dissipation terms for the velocity form of shallow-water equations as well as vorticity–Montgomery potential form are similar, except that there is a factor of $1/H$ in front of each expression (4)–(6).

In the case of uniform layer thickness and nondivergent flow, the difference between all four versions disappears and all of them have properties (i)–(iv), which makes the simplest possible version, Laplacian of velocity, physically correct. This situation is identical to incompressible two-dimensional hydrodynamics.

In the general case, however, none of the three versions of (4)–(6) satisfies all four requirements (i)–(iv). Formal vertical integration of the three-dimensional primitive equations with depth-independent horizontal velocities leads to (4) (Gent 1993). This term is consistent with (iv) but not with (i), (ii), and (iii). Form (5) is in agreement with (i), (ii), and (iv) but violates (ii); while (6) does not satisfy (ii), (iii), and (iv).

4. Consequences of different forms of dissipation

Despite being commonly used in ocean modeling, dissipation operators may cause severe violations of the physical principles. Many models successfully use one of the forms (4)–(6) without producing physically unrealistic results. In this section we shall analyze several typical geophysical situations in order to evaluate the importance of the use of a physically consistent formulation of lateral dissipation.

a. Potential flow

In the context of a nonrotating environment and vorticity-free flow, that is, $\partial u_\mu/\partial x_\beta = \partial u_\beta/\partial x_\mu$, the version (5) of the friction term can be interpreted as the divergence of a symmetric tensor and is consistent with all four requirements (i)–(iv). Moreover, if no vorticity is created on the boundary or brought through the boundary (e.g., in the case of appropriate influx boundary condition), the flow remains vorticity free even though it is a viscous flow. However, the realization of this situation requires a very special configuration of stress-free boundaries or the consideration of boundless flow.

b. Midlatitude mesoscale vortex dynamics

For a flow close to the quasigeostrophic regime, the disturbance of layer thickness is small. In these types of flows, the largest gradients of the layer thickness are associated with coherent structures and are advected with the fluid particles. Consequently, for a given fluid particle the layer thickness tends to be conserved as long as the coherent structure remains the same. The characteristic timescale of the structural change of the flow is related to the timescale associated with the interaction of different coherent structures rather than advection of particles within the same coherent structure. This timescale is relatively large, which makes the flow practically nondivergent. The divergence of the velocity field is of order of $Ro(\Delta t/\ell) \ll 1$ relative to the deformation rate, so the Laplacian of the velocity becomes a reasonable approximation for the friction term. Term $Ro \ll 1$ is the Rossby number and $\Delta t/\ell \ll 1$ is the ratio of the vertical and horizontal scales, where $\ell$ is the characteristic horizontal scale associated with
coherent structure. Using the midlatitudinal deformation radius as the characteristic horizontal scale, $L = (g'\alpha)^{1/2} f^{-1}$, one can easily estimate

$$\frac{\mathcal{H}}{L} = \frac{ug}{fL^2} = \frac{\mathcal{I}}{g'} = \frac{1 \times 10^{-4}}{3 \times 10^{-2}} = 3 \times 10^{-3}.$$  

Indeed, this estimation gives an even better justification for the use of this form of friction than the range of applicability of the quasigeostrophic theory itself.

c. Linear equatorial dynamics

The disturbance of the layer thickness is small because of linearity. Moreover, from the numerical point of view, linear models do not cascade enstrophy to the underresolved scales, and, therefore, require a relatively small dissipation. We should not expect large distortion produced by dissipation terms (4)–(6).  

d. Nonlinear equatorial dynamics

Some equatorial motions (e.g., Yanai waves) provide a mechanism for sufficient change of relative vorticity and divergence of the velocity field at the same time. Neither potential nature of the flow nor nondivergence associated with the geostrophic balance are present. None of the speculations presented above is possible in this case. The use of a physically consistent formulation of lateral dissipation is desirable.

e. Flow over topography

In this case the variation of the layer thickness is not small, even if the fluid is nearly at rest, for example, barotropic flow over a sloping bottom or for the bottom layer of stratified multilayer system. In both situations, the divergence of the velocity field is of order of $O(\nabla H)$ of the value of deformation rate, except for the case of linear geostrophy on an $f$ plane when the fluid motion exactly follows the isobates. In the next section we demonstrate that the use of nonphysical dissipation leads to a considerable discrepancy of the results using different versions of the viscous dissipation, even in the case when the friction coefficient is kept as small as possible.

5. Barotropic flow over topography:

A computational example

As seen from the discussion above, friction terms used in numerical models do not always agree with the postulates (i)–(iv). Thus, as a consequence from (ii), the asymmetry of the viscous stress tensor cannot be physically small. However, because the size of the fluid element cannot be smaller than the grid size $\Delta x$, in a numerical model tuned to the minimum possible dissipation, the infinite vorticity sources may alias as finite sources. This is because for the second-order finite-difference approximation, the amplitude of the numerical noise decreases as $(\Delta x)^2$, the viscous dissipation used for suppression of the noise is reduced in the same way. Thus, the artificial source of vorticity remains finite.

In the numerical experiment described below, we simulate barotropic shallow-water flow over steep topography. Three versions of viscous dissipation were tested: the trace-free version of (3) and viscous terms (5) and (6). The assumption $A_2 = 0$ in (3) is not restrictive and our approach can be extended to the general case. Schär and Smith (1993) discuss the properties of physically correct friction operator (3) but implemented terms of form (5) into the numerical model (see appendix B to Schär and Smith 1993 for details). Dissipation term (5) is used in Bleck and Boudra's (1981) isopycnic coordinate model, while (6) is used by Oberhuber (1993) and Killworth et al. (1991) for the free-surface barotropic mode.

In all three cases, the friction coefficients were equal to each other and were chosen to be as small as possible, but sufficient to maintain the numerical stability of the model. The Reynolds number for the experiments described below (based on the domain size) is roughly 12 000, grid resolution—301 × 301. We employ a fully implicit barotropic shallow-water equation model, which can be used for both compressible (with free surface) and incompressible (rigid-lid approximation) flows. Crank–Nicholson in time-marching scheme is used. Spatial discretization the momentum and continuity equations is similar to those described is the next section, except that Cartesian instead of spherical coordinates are used.

The concentric dashed circles on Fig. 1 represent a Gaussian "sea mountain." The height of the top of the mountain equals 90% of the layer depth away from the mountain. We initialize a dipole away from the mountain by specifying the desirable potential vorticity (PV) field and resolving the coupled nonlinear elliptic problem to obtain the transport streamfunction (the initial state is assumed to be nondivergent) and initial layer thickness. See appendix A for details. This initial condition allows a smooth start without exciting strong surface gravity waves. The initial PV is uniform everywhere except for the dipole, which consists of two eddies of equal intensity (in terms of relative vorticity). Both eddies are originally circular and have roughly uniform PV distribution within their cores with steep fronts on the edges of the cores. These experiments are performed on the $f$ plane. In order to visualize the mo-
Friction term in form of divergence of the tensor Eq. (3) is used. Reynolds number, based on domain size is $10^4$. (b) The same as (a) but friction term given by Eq. (5).

Figure 1a shows the evolution of PV for the case of physically correct dissipation (3). The dipole approaches to the mountain, interacts with it, turns, leaves it, and moves away along the straight line in a direction approximately 170° to the right from the direction of the original propagation. After the interaction, the original shape of the dipole is restored and there is a small tail of vorticity behind it, which slowly disappears being advected by the background flow and eventually dissipated by friction. It should be pointed out that even if there is no local PV anomaly, water is never at rest. When the dipole is far away from the mountain (before or after interaction), there is weak clockwise circulation of water above the mountain. This is necessary to compensate the $f/H$ term in PV in order to keep it uniform. This circulation becomes visible by the slow advection of the vorticity filament left after the dipole leaves the mountain. Since the velocity field associated with this flow does not correspond to the solid-body motion, the friction causes some modification of the flow. Eventually PV will not be uniform near the top of the mountain, even for the case of the physically consistent friction term, however, in the experiments presented here the friction coefficient is kept sufficiently small, so the characteristic timescale of this modification is much larger than the interaction time. Thus, we should not expect departure from the uniform PV near the top of the mountains in the experiment with physically consistent friction. Note that during the whole interaction process, the values of the minimum and the maximum of PV anomalies inside the cores remain the same, and the only visible consequence of dissipation is the smoothing of the fronts on the edges of the vortex cores.

The content of Fig. 1b is the same as Fig. 1a, but friction is now given by (5). The behavior of PV is approximately the same, but an artificial generation of PV is seen near the top of the mountain. Note that before the dipole approaches the top of the mountain, the spot of negative vorticity has been created. The dipole picks the spot and carries it away from the mountain, so that no PV anomaly is seen on the top of the mountain after the dipole left the top. But eventually a new spot develops and linear growth of the PV anomaly is seen on the top of the mountain.
There is an artificial generation of PV if friction operator (6) is used (Fig. 1c).

The difference between the fields of PV in Figs. 1b and 1a is shown in Fig. 1d. Figure 1e represents the difference between Figs. 1c and 1a. The contour interval in Figs. 1c and 1d is ten times finer than in Figs. 1a–c. White areas inside the most dark pattern correspond to the region where the difference between PV anomalies is larger than 20%.

Comparison of Figs. 1a, 1b, and 1c shows that there are two consequences of artificial sources of vorticity. First, the residual flow on the top of the mountain. Second, the deflection angle of the dipole is slightly changed. The second effect is very small and it cannot be seen in Figs. 1a–c. But it explains the relatively large difference of PV fields near the dipole in Figs. 1c and 1d at \( t = 2.00 \). It is interesting to note that this change has opposite signs for the dissipation terms (5) and (6).

This experiment demonstrates that if the friction term (3) is used, the result is very similar to inviscid theory (see Kloosterziel and Carnevale 1993; Flör 1994 for a review). The friction operator we used provides effective damping of the numerical noise without additional effects. We have investigated all three dissipation operators (3), (5), (6) for the case of compressible (with free surface) and incompressible (with rigid lid) barotropic shallow-water flows. The numerical model we used provides a possibility of gradual transformation from the compressible to the incompressible case. The second may be obtained as the limit of the first when the acceleration of gravity goes to infinity. In this case, the ratio of the elevation of free surface to the layer thickness becomes infinitely small. This ratio may be interpreted as a natural measure of compressibility of the flow. In “the most compressible case” we have considered the maximum value of this parameter may be of order of 0.5 when the dipole reaches the top of the mountain.

It was found that the friction operator (3) does not produce artificial sources of vorticity. Friction term (5) gives a tolerable solution with relatively small artificial sources of vorticity for the incompressible case, but the departure from the incompressibility causes dramatic increase of the artificial generation of vorticity.

6. Boundary conditions in the numerical ocean models

We have discussed physical and numerical consequences of dissipation in the interior of the basin. Besides its physical role, dissipation controls the stability...
of numerical models. In many cases of ocean modeling the coordinate lines cannot be aligned with the coastlines because the shape of the basin is often too complex. As usual the coastline is represented by straight-line segments with sharp corners, which leads to loss of accuracy of approximation. Because of that, the numerical instability often appears first not inside the model basin but on the boundary. The necessity to eliminate this instability requires a larger dissipation coefficient. As a rule, this leads to a global increase of dissipation so that interesting features of the fluid motion may be suppressed.

Slip-free boundary conditions are often implemented as reflection boundary conditions. On a staggered Arakawa C grid, the velocity component normal to the boundary is explicitly set to zero, while the tangential velocity is reflected with respect to the wall. Then the interior discretization scheme is applied. This approach has two shortcomings: First, one can easily see (appendix B) that it leads to an underestimated of the viscous term near the boundary. Second, the reflection rule conflicts with no normal component boundary condition on convex corners of the coastline as well as on small islands, represented by one or two grid points. The no-flux boundary condition always overrides the reflection and it can be shown that the friction term is approximated in the points in front of the corner with the same truncation error as in interior. If the coastal line has a region where corner points interleave with short segments of reflection boundaries, this place behaves as a source of numerical distortion because terms are approximated with different accuracy in adjacent points. This situation often occurs when coastline goes in approximately diagonal direction with respect to grid cells. In this section we describe an alternative implementation of no-slip boundary conditions, which is free of the above disadvantages.

The nonlinear and viscous stress terms of the shallow water equations can be written as the divergence of a symmetric momentum flux tensor, so in the geophysical spherical coordinates shallow-water equations have the form:

$$\frac{\partial V_\lambda}{\partial t} + \nabla P_{\lambda\phi} + \nabla P_{\lambda\phi} - \frac{2\tan\phi}{a} P_{\lambda\phi} - fV_\phi = -\xi_\lambda + \eta_\lambda$$

$$\frac{\partial V_\phi}{\partial t} + \nabla P_{\lambda\phi} + \nabla P_{\lambda\phi} + \frac{\tan\phi}{a} (P_{\lambda\phi} - P_{\phi\phi})$$

$$+ fV_\lambda = -\xi_\phi + \eta_\phi$$

$$\frac{\partial H}{\partial t} + \nabla V_\lambda + \nabla V_\phi - \frac{\tan\phi}{a} V_\phi = 0,$$  \hspace{1cm} (7)

where \(\lambda, \phi\) are longitude and latitude. The differential operators \(\nabla_\lambda\) and \(\nabla_\phi\) and their commutation rule are defined as follows:

$$\nabla_\lambda = \frac{1}{a \cos \phi} \frac{\partial}{\partial \lambda}, \quad \nabla_\phi = \frac{1}{a} \frac{\partial}{\partial \phi},$$

so that

$$\nabla_\lambda \nabla_\phi = \nabla_\phi \nabla_\lambda + \frac{\tan \phi}{a} \nabla_\lambda.$$  \hspace{1cm} (8)

Here \(H\) is the layer thickness; \(V_\lambda, V_\phi\) are transports in the longitudinal and meridional directions; \(f = 2\Omega \sin \phi\) is the Coriolis parameter; \(\xi_\lambda, \xi_\phi\) and \(\eta_\lambda, \eta_\phi\) are the pressure gradient terms; and \(\xi_\lambda, \xi_\phi\) and \(\eta_\lambda, \eta_\phi\) are the external forces: wind stress, effect of the bottom topography, etc. Term \(P\) is the symmetric tensor of the momentum flux density:

$$P_{\lambda\lambda} = V_\lambda V_\lambda - A \sigma_{\lambda\lambda}, \quad P_{\lambda\phi} = V_\lambda V_\phi - A \sigma_{\lambda\phi},$$

$$P_{\phi\phi} = P_{\phi\phi} = \frac{1}{2} (V_\phi V_\phi + V_\lambda V_\lambda) - A \sigma_{\phi\phi}. \hspace{1cm} (9)$$

Here \(A\) is the horizontal (strictly speaking, isopycnal) friction coefficient and \(\sigma_{\lambda\phi}\) is the deformation tensor. This tensor is symmetric and trace-free; so

$$\sigma_{\lambda\lambda} = -\sigma_{\phi\phi} = V_\lambda V_\lambda - V_\phi V_\phi - \frac{\tan \phi}{a} V_\phi,$$

$$\sigma_{\phi\phi} = V_\phi V_\phi = V_\lambda V_\lambda - \frac{\tan \phi}{a} V_\lambda.$$
Physically consistent dissipation can be cast into the same form as the nonlinear part, and therefore, we may skip a large amount of computational work by computing the viscous stress tensor and add it to the nonlinear stress tensor, which will be used then to compute both nonlinear and viscous terms in the momentum equations. It should be noticed that if one would try to compute the viscous terms directly and retain all geometrical terms in the case of spherical geometry, it will lead to an unefficient code because even the simplest friction operator, say Laplacian of velocity vector, is very complex. In the proposed approach, the second derivatives of velocities will be almost never computed explicitly, consequently the dissipation term will require a relatively small fraction of the computations. We will see that spherical geometry introduces only a small additional amount of computations in comparison with Cartesian coordinates for the case when the tensor formalism is retained in the discrete model.

The dissipation operator implemented this way does not disturb the symmetries of the original inviscid equations not only in continuous but also in the discrete formulation. To some extent, this idea is similar to Arakawa and Hsu (1990), where the dissipation was build into the discrete scheme of the nonlinear terms.

The geometrical terms in the spherical coordinate momentum equations inhibit some numerical algorithms that may be successfully used for Cartesian coordinates. In particular, we will show that in most cases land boundary conditions cannot be formulated as a reflection rule for the tangential velocity on the north and south coastlines.

As the present approach may be applied for different problems, we do not specify the particular form of the pressure gradient terms. Equations (7) may be used in context of a barotropic single-layer model or may refer to an isopycnic layer in a multilayer model. The form of the dissipation terms does not depend on the application.

An Arakawa C grid is used to discretize (7). In our elementary stencil $V_{j,k}$ is located one-half grid to the south relative to the $H_{j,k}$, and $V_{j+1,k}$ is one-half grid east from the $H_{j,k}$ point. The grid intervals $\Delta \lambda$ and $\Delta \phi$ are the distances between alike points, for instance, $V_{j,k}$ and $V_{j+1,k}$:

$$V_{j,k+1} \rightarrow H_{j,k} \leftarrow V_{j+1,k} \rightarrow H_{j+1,k}$$

$$V_{j,k} \uparrow V_{j+1,k}$$

$$V_{j+1,k-1} \rightarrow H_{j+1,k} \leftarrow V_{j+1,k-1} \rightarrow H_{j,k-1}.$$  \hspace{1cm} (11)

Centered differences are used to approximate spatial derivatives of all fields.

Denote

$$\Gamma = \frac{\tan \phi}{a}$$  \hspace{1cm} (12)

Introduce the average and finite-difference operators,

$$\bar{q} = \frac{q(\lambda + \Delta \lambda/2, \phi) + q(\lambda - \Delta \lambda/2, \phi)}{2}$$

$$\bar{q}^\phi = \frac{q(\lambda, \phi + \Delta \phi/2) + q(\lambda, \phi - \Delta \phi/2)}{2}$$

$$\delta q = \frac{q(\lambda + \Delta \lambda/2, \phi) - q(\lambda - \Delta \lambda/2, \phi)}{\Delta \lambda \cos \phi}$$

$$\delta \phi q = \frac{q(\lambda, \phi + \Delta \phi/2) - q(\lambda, \phi - \Delta \phi/2)}{a \Delta \phi}.$$  \hspace{1cm} (13)

If the diagonal elements of the tensor $\mathbf{P}$ are specified at $H$ points and off-diagonal at the vorticity points (hereafter referred as $q$ point) staggered one-half grid east and one-half grid south from the respective $H$ point), the discretization of (7) is straightforward:

$$\frac{\partial V_{\lambda}}{\partial \tau} + \delta_\lambda (P_{\lambda\lambda}) + \delta_\phi (P_{\phi\phi}) - 2 \Gamma \bar{P}^\phi = - \bar{p}_\lambda + \bar{q}_\lambda$$

$$\frac{\partial V_{\phi}}{\partial \tau} + \delta_\lambda (P_{\lambda\phi}) + \delta_\phi (P_{\phi\phi}) + \Gamma (\bar{P}_{\lambda\phi} - \bar{P}_{\phi\phi}) + f \bar{V}^\phi = - \bar{p}_\phi + \bar{q}_\phi$$

$$\frac{\partial H}{\partial \tau} + \delta_\lambda V_{\lambda} + \delta_\phi V_{\phi} - \Gamma \bar{V}^\phi = 0,$$  \hspace{1cm} (14)

where the components of the momentum flux tensor are given by

$$P_{\lambda\lambda} = \bar{V}_{\lambda} \bar{v}_{\lambda} - \Delta \lambda (\delta_\lambda V_{\lambda} - \delta_\phi V_{\phi} - \Gamma \bar{V}^\phi)$$

$$P_{\phi\phi} = \bar{V}_{\phi} \bar{v}_{\phi} + \Delta \phi (\delta_\lambda V_{\lambda} - \delta_\phi V_{\phi} - \Gamma \bar{V}^\phi)$$

$$P_{\lambda\phi} = P_{\phi\lambda} = \frac{1}{2} (\bar{V}_{\lambda} \bar{v}_{\phi} + \bar{V}_{\phi} \bar{v}_{\lambda}) - \Delta \lambda \Delta \phi \bar{v}_{\lambda} \bar{v}_{\phi} - \Gamma \bar{V}^\phi,$$  \hspace{1cm} (15)

and the velocities $v_{\lambda}$ and $v_{\phi}$ are computed from the mass fluxes $V_{\lambda}$ and $V_{\phi}$ according to

$$v_{\lambda} = \frac{V_{\lambda}}{H}, \hspace{0.5cm} v_{\phi} = \frac{V_{\phi}}{H}.$$  \hspace{1cm} (16)

Equations (14)–(16) correspond to the symmetrized version of Lilly discretization scheme for the nonlinear terms (Lilly 1965). Note that in general $\bar{V}_{\lambda} \bar{v}_{\lambda} = \bar{V}_{\phi} \bar{v}_{\phi}$ because of the staggered grid. In (15) both the nonlinear and the viscous part retain the symmetry $P_{\lambda\phi} = P_{\phi\lambda}$.

The discrete equations (14) are valid everywhere, including the points near the boundaries. However, expressions (15) are applicable for the interior grid points.
of the domain only. Thus, a special discretization of the stress tensor $P_{ad}$ is required on and near the boundaries.

Before discussing the discrete version of the land boundary conditions, we should note that (14), along with (15) and (16), gives a second order of accuracy $O[(a\Delta \lambda \cos \phi)^2 + (a\Delta \phi)^2]$ approximation for the set of (7). Equations (15) give the second order of accuracy approximations for the components of $P_{ad}$. After substitution of these expressions into (14), the last ones should give the first order of accuracy approximation of (7) because of division by $a\Delta \lambda \cos \phi$ and $a\Delta \phi$ in finite differencing. The reason why there is no actual loss of the order of accuracy is the error cancellation due to the centered-difference scheme. Errors of interpolation and finite-difference errors of approximation of $P_{ad}$, calculated in two adjacent grid points according to (15), may be estimated as the second derivatives of some functions times quadratic expressions of $\Delta \lambda$ and $\Delta \phi$. For the centered-difference scheme, these errors will cancel each other and the residual error occurs only in the next order of accuracy. This means that for the boundary points when one-sided finite-difference approximations are used to calculate derivatives, these approximations must be designed in such a way that they simulate not only the derivative of the function but also the truncation error of this derivative as it would be if it is approximated by the centered-difference scheme. If this rule is always satisfied, the result will be the complete second order of accuracy approximation for (7) and the boundary conditions.

This idea may be illustrated by the following example. If $f(x)$ is some function of coordinate $x$, $f_j = f(x_j)$, $\Delta x = x_{j+1} - x_j$ and the centered-difference scheme is used to approximate the derivative of $f$,

$$\frac{\partial f}{\partial x} \bigg|_{x_j} \approx \frac{f_{j+1} - f_{j-1}}{2\Delta x},$$

and it is easy to see that

$$\frac{f_{j+1} - f_{j-1}}{2\Delta x} = \frac{\partial f}{\partial x} + \frac{1}{6} \frac{\partial^3 f}{\partial x^3} (\Delta x)^2 + O((\Delta x)^3),$$

and therefore, when using one-sided finite-difference approximation for the derivative of $f$, it should be done as follows:

$$\frac{\partial f}{\partial x} \bigg|_{x_j} \approx \frac{\partial f}{\partial x} \bigg|_{x_j} + \frac{1}{6} \frac{\partial^3 f}{\partial x^3} \bigg|_{x_j} (\Delta x)^2.$$

This may be also rewritten as

$$\frac{\partial f}{\partial x} \bigg|_{x_j} = \frac{f_{j+1} - f_{j-1}}{2\Delta x}.$$

where

$$f_j^{*} = 4f_j - 6f_{j-1} + 4f_{j-2} - f_{j-3}. \quad (17)$$

The right expression is nothing but the fourth order extrapolation for $f_j^{*}$.

In the approach we have accepted, the coastline is represented by segments of straight lines. One can identify eight types of segments, four in longitudinal and meridional directions (90°): that is, north, east, south, and west coasts of the ocean; the other four are diagonal (45°) segments: northwest, northeast, southeast, and southwest coasts of the ocean. These names are given by direction of a vector normal to the coastline and pointed from water to land. No-slip boundary conditions are implied on all solid boundaries. In the configuration adopted here, meridional boundaries pass through $V_\lambda$ points; zonal pass through $V_\phi$ points, while all diagonal boundaries are the straight lines going through both $V_\lambda$ and $V_\phi$ points. After the model land mask is set up from the model topography data, one can identify and store in special arrays indices of 12 groups of coastline boundary points: east and west $u$-boundary points (2); north and south $v$ points (2); 90° $q$-boundary points, where special treatment is needed to compute viscous stress part of the off-diagonal components $P_{u\phi}$ (4); and 45° $q$-boundary points, also for $P_{u\phi}$ (4).

Equation (16) give the right values of the velocities everywhere, including the boundary points, despite of the fact that the land values of $H$ points are accidentally used: the velocities are equal to zero at the land boundary points anyway.

No special $H$ points are needed at this time: the diagonal components $P_{\lambda\lambda}$ and $P_{\phi\phi}$, defined at $H$ points, are not require a special treatment near the boundaries.

Since both components of velocity vanish on the coastline, the only remaining terms of the off-diagonal elements $P_{u\phi}$ on the 90° boundaries are

$$P_{u\phi} = -AH\nabla\phi v_\phi \begin{cases} \text{on the western} \\ \text{and eastern coasts} \end{cases}$$

and

$$P_{u\phi} = -AH\nabla\lambda v_\lambda \begin{cases} \text{on the northern} \\ \text{and southern coasts}. \end{cases}$$

In the interior of the domain, the centered-difference expression is used to estimate $\nabla\lambda v_\phi$ at the $q$ points:

$$\nabla\lambda v_\phi \bigg|_{l+1/2,k} \approx \delta_\lambda v_\phi = \frac{1}{a \cos \phi} \left\{ \frac{\partial v_\phi}{\partial \lambda} \bigg|_{l+1/2,k} + \frac{1}{6} \frac{\partial^3 v_\phi}{\partial \lambda^3} \bigg|_{l+1/2,k} (\Delta \lambda)^2 + O((\Delta \lambda)^3) \right\} \times \left\{ \frac{\partial v_\phi}{\partial \lambda} \bigg|_{l+1/2,k} + \frac{1}{6} \frac{\partial^3 v_\phi}{\partial \lambda^3} \bigg|_{l+1/2,k} (\Delta \lambda)^2 + O((\Delta \lambda)^3) \right\}$$

Implying that $v_\phi \big|_{l+1/2,k} = 0$, and using three interior grid points next to the boundary, one can construct the one-sided finite-difference approximation for $\nabla\lambda v_\phi$, which
gives the same truncation error in the second order of accuracy as the expression above,

$$\nabla \vec{v}_j|_{j+1/2,k} \approx \frac{-4v_{j+1,k} + v_{j-1,k} - \frac{1}{5} v_{j+2,k}}{a \Delta \lambda \cos \phi_{j+1/2,k}}$$

$$= \frac{1}{a \cos \phi} \left\{ \frac{1}{2} \frac{\partial^2 v_j}{\partial \lambda^2} + \frac{1}{6} \frac{\partial^3 v_j}{\partial \lambda^3} \right\} \left( \frac{\Delta \lambda}{2} \right)^2$$

$$+ O((\Delta \lambda)^2) \quad 18$$

Boundary condition (18) can be also formulated as a reflection rule for the tangential velocity:

$$v_{j+1/2,k} = -3v_{j+1/2,k} + v_{j+1/2,k} - \frac{1}{5} v_{j+2/2,k} \quad 19$$

which is the fourth-order extrapolation scheme based on the values of the function at points $j + 1/2, k; j, k$; $j - 1, k; j - 2, k$, with the implicit condition that $v_{j+1/2,k} = 0$. After all $v_{j+1/2,k}$ has been reflected around the boundary, the centered scheme (15) can be formally applied to compute the off-diagonal components of the stress tensor $P$. Despite $v_{j+1/2,k} + v_{j-1/2,k} = 0$, no error will be made in the calculation of the nonlinear term because the other component of the velocity field as well as transport vanishes anyway. However, one can verify that in the geometrical term in viscous part of (15) will be misrepresented if the reflection rule similar to (19) will be used at northern or southern boundary.

Despite the fact that the stencil used by (18) and (19) is two grid points wider that those of the traditional reflection rule, only one extra grid point will be involved into finite-difference scheme for the momentum equations. This is because the additional points are always taken from the offshore side, and the off-diagonal element computed according to (18) interacts with its neighbor one grid point in the offshore direction. So only one of the two extra points will not overlap. In situations when (18) is used in a two-gridpoint-wide channel, the remaining extra point may be accidentally taken from land of the opposite side of the channel. No problem occurs, if the value of the velocity component in this land grid point is zero. One can easily verify that no error will be made in calculation of the integrated mass flux through the channel in comparison with the exact solution. Moreover, even for a one-gridpoint-wide channel, where two of the three velocity points in (18) are taken from land, the net mass flux will be overestimated in 1.5 times, which is still better than the standard reflection rule, which leads to overestimation in 3 times. However, problems may occur if the reflection rule (19) is used instead of (18) in a narrow straight: the land velocity points may have nonzero values because they themselves may be reflected ghost points. Thus, we recommend use of the reflection formulation (18) only for a simple-shaped domain. From this point of view (18) is much more robust.

Approximation (15) for $P_{jk}$ components defined at $q$ points requires interpolation of the layer thickness to $q$ points. This interpolation cannot be performed for $P_{jk}$ elements located at 90° land boundaries or $P_{jk}$ components adjacent to 45° boundaries, because it would require the values of layer thickness located at land points. Therefore, an extrapolation is needed to evaluate layer thickness on the boundary.

In the case of barotropic incompressible flow over topography, these values of $H$ at land points are externally predetermined and may be extrapolated from the interior of the domain.

In the compressible case we assume that there is an equation of state, which hydrostatically relates pressures of isopycnic layers $l = 1, \ldots, L$ of a multilayer system with the layer thicknesses $H^{(l)}$,

$$p^{(l)} = \Phi^{(l)}(H^{(1)}, \ldots, H^{(L)}, D), \quad l = 1, \ldots, L, \quad 20$$

where $D$ represents the bottom topography. We assume that (20) may be always resolved so that

$$H^{(l)} = \Phi^{(l)}(p^{(1)}, \ldots, p^{(L)}, D), \quad l = 1, \ldots, L. \quad 21$$

As an immediate consequence from (20),

$$\nabla p^{(l)} = \frac{\partial \Phi^{(l)}}{\partial H^{(l)}} \nabla H^{(l)}. \quad 22$$

On the other hand, formal application of the momentum (7) to the boundary grid point leads to the boundary condition for the pressure field. For example, the $V_z$-component momentum equation applied to the eastern coast yields

$$-\nabla \cdot (AH \nabla v_0) - \nabla \cdot (AH \sigma_{\omega 0})$$

$$+ 2\Gamma A H \sigma_{\omega 0} = -H \nabla \cdot p + \mathcal{S}_z, \quad 23$$

where $H$, $v_0$, $\sigma_{\omega 0}$, and $\mathcal{S}_z$ are layer thickness, velocity, pressure, off-diagonal component of deformation tensor, and volume forcing (wind, interfacial drag, etc.) of an isopycnic layer. To derive (23) we have used the assumption that both component of velocity vanish on the boundary. We have dropped superscript ($l$) for the simplicity of notation. Since $\sigma_{\omega 0}$ can be computed on the boundary points according to (18), discretization of $\nabla \cdot (AH \sigma_{\omega 0})$ is obvious. A second order of accuracy one-sided finite-difference approximation $s$ needed to compute $\nabla \cdot (AH \nabla v_0)$:

$$\nabla \cdot (AH \nabla v_0)_{j,k} \approx \frac{A}{2} \left( \frac{H_{j,k} + H_{j+1,k}^s}{2} \right)$$

$$- \frac{2v_{j,k} - 5v_{j-1,k} + 4v_{j-2,k} - v_{j-3,k}}{(a \Delta \lambda \cos \phi)^2} + A \frac{H_{j+1,k}^s - H_{j,k}^s}{a \Delta \lambda \cos \phi}$$

$$+ \frac{11}{6} \frac{v_{j+1,k} - 3v_{j-1,k} + 2v_{j-2,k} - v_{j-3,k}}{a \Delta \lambda \cos \phi} \quad 24$$
FIG. 2. (a)–(d) Comparison between the new implementation of no-slip boundary conditions and the traditional method of reflections. Field plotted in contours is the potential vorticity field computed from output of 0.25° resolution 1.5-layer reduced gravity model of equatorial Atlantic. Only western part of the model domain, coast of Brazil, is shown. Arrows represent velocities. Every second vector in each direction is plotted on the top two panels, while bottom panels contain all vectors. (a) Method of implementation of the no-slip boundary conditions from the present section is used. Viscous coefficient is 100 m² s⁻¹. (b) An enlarged, more detailed plot of the most dynamically active part of coastal region shown in Fig. 2a. (c), (d). The same as (a), (b), but the traditional method of reflection ghost points is used for the boundaries. Viscosity is 200 m² s⁻¹, which is nearly on the edge of the numerical stability.

where $H_{j+1}^*$ is the extrapolated value of layer thickness and $u_{w, k} = 0$ on the boundary.

Equation (23), along with (22) and (21), forms the boundary condition for the normal derivative of pressure and finally gives the extrapolation rule for layer thickness.

To some extent, the necessity of the boundary conditions for pressure is a mathematical artifact because of the use of a staggered grid. From the discussion above, we see that these boundary conditions emerge from the necessity to extrapolate the layer thickness to the land points for the computation of the viscous part of the stress tensor. Physical meaning of (23) is the balance between lateral friction, pressure gradient, and external forces. This is a diagnostic relationship. Some shallow-water ocean models (e.g., Hurlburt and
Thompson 1980) explicitly form and solve the pressure Poisson equation according to the split-implicit time-stepping scheme they use (Kwizak and Robert 1971). In this approach, the friction term is lagged in time to satisfy the numerical stability criterion, and therefore the left-hand side of (23) and the forcing term taken from the previous time step, thus, (23) plays the role of the boundary condition for the elliptic problem for the pressure field. After it is solved, the land values of layer thickness may be extrapolated according to (22) or (21) and used to compute friction terms at the next time step.

It is numerically advantageous to apply the friction operator at the most new time step by using backward Euler time step for the explicit dissipation. A time-splitting scheme may be employed to implement (22) – (23) into an explicit model.

In the general case (22) – (23) form a nonlinear problem, which may be solved iteratively. Some models use fully implicit discretization in time and solve the momentum and continuity equations simultaneously instead of solving the elliptic problem for pressure (e.g., Oberhuber 1993; see also Gresno and Sani 1987 and references therein; Tan Weiyan 1992 for review). As usual this is done by an iterative procedure. Equation (22) – (23) may be naturally incorporated into this procedure, so that (22) – (23) will be satisfied at any time step.

When a rigid-lid approximation is applied to a multilayer system, (20) changes to

\[ p^{(n)} = p_s + \Phi^{(n)}[H^{(1)}, \ldots, H^{(L)}, D], \]

where \( p_s \) is the barotropic part of the pressure, which is no longer a function of state. The incompressibility assumption implies that

\[ \sum_{i=1}^{L} H^{(i)} = D = \text{const} \]

\[ \sum_{i=1}^{L} \rho^{(i)} H^{(i)} = \text{const} \]

where \( D \) and \( \rho^{(i)} H^{(i)} \) are the barotropic free surface version and baroclinic free surface version, respectively.

In the conclusion of this section, we shall compare two ocean models with different versions of the numerical formulation of the no-slip boundary conditions, as described by Blanke and Delecluse (1993), Inoue and Welsh (1993), and other models with similar grid resolution, and it is capable to remove most of the computational noise in Figs. 2c,d.
on a realistic form of coastline. Both models are 1.5-layer reduced gravity models. A fully implicit two-time level time-marching scheme is used. Backward Euler time step is used for dissipation terms, while all other terms are discretized according to second-order accurate in time Crank–Nicholson scheme. This model remains numerically stable for relatively low, in comparison with an explicit version, values of the viscous dissipation. This allows one to obtain the numerical solutions for a wider range of dissipation coefficient for both versions of boundary conditions. Grid resolution is 0.25° in both longitudinal and latitudinal directions.

The field shown on Fig. 2a is the potential vorticity field (contours) and velocity (arrows) computed from model, which used the implementation of the boundary conditions described in this section. Fragment of the model domain with Brazilian coast is shown. Viscous coefficient is 100 m² s⁻¹. In order to make these figures, velocity components are linearly interpolated to the vorticity points. Every second vector in each direction is plotted of Fig. 2a, while all vectors are plotted on Fig. 2b, which shows the fragment of Fig. 2a in more details. Despite the relatively low dissipation, there is practically no computational noise on these two figures.

In contrast to Figs. 2a, b, fields on Figs. 2c, d, where the traditional reflection boundary conditions are used, are much noisier, although all features of the large-scale field are retained. Viscosity is 200 m² s⁻¹. The conventional reflection boundary model suffers lack of dissipation along the 90° boundaries, which causes conflict with the accuracy of representation of the viscous stress at 45° boundaries. Fluid artificially accelerates along the 90° boundary and suddenly brakes at the conjunction point with 45° boundary. As a consequence, these corners generate grid-scale noise on the vorticity field (Fig. 2d). The computational noise may be suppressed by the increased dissipation (Fig. 2e), where A = 400 m² s⁻¹ and (Fig. 2f) A = 700 m² s⁻¹. The final value of A = 700 m² s⁻¹ is comparable with dissipation coefficients used by other models with similar grid resolution (Blanke and Delecluse 1993; Inoue and Welsh 1993; see also literature referred there). On Fig. 2f the noise almost disappears, but at the same time one can see a significant reduction of the dynamic activity of the boundary layer. As a result, we can see that the new method of discretization of the no-slip boundaries allows to decrease viscous dissipation necessary for the numerical stability.

7. Summary

We have discussed different forms of dissipation terms used in numerical ocean modeling, focusing our attention to the dual purpose of the presence of dissipation in numerical models.

The physically consistent form of the dissipation term in shallow-water equations cannot be derived from the dissipation term in the primitive equations. We have formulated the viscous dissipation term that preserves the integral constraints derived in the general theory of stratified flows.

A new approach of the numerical implementation of the no-slip boundary condition for a realistic coastline was developed. This approach explicitly uses the intrinsic tensor structure of the momentum equations in flux form both in the interior of the domain and on the boundaries. Thus, instead of discretizing boundary conditions for the velocities, we discretize the components of the stress tensor on the boundaries. This discretization is designed to approximate not only the components of the stress tensor on the boundaries but also the truncation error of the numerical scheme used in the interior of the domain.

It was demonstrated that the new approach provides a more accurate approximation of the equations of motion near the boundaries, which reduces the undesirable effects of dissipation and obtains a more valuable numerical solution.

Acknowledgments. COAPS receives its base support from the Secretary of NAVY Research Grant N00014-85-J-1240 from the Office of Naval Research. This research is supported by the Advanced Research Projects Agency Grant SC25046 managed by Dr. Ralph Aline and the Strategic Environmental Research Development Program (SERDP) headed by Dr. John Harrison. Special thanks to Dr. Detlev Müller, Dr. Steve Meyers, and Dr. Mark Johnson for their comments and fruitful discussion.

APPENDIX A

Generation of the Initial State for the Shallow-Water Flow

Assume that the flow is initially nondivergent, therefore, the transports U and V can be expressed in terms of the streamfunction ψ,

$$U = -\frac{\partial \psi}{\partial y}, \quad V = \frac{\partial \psi}{\partial x}$$

so that

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0.$$ 

The potential vorticity field is given by

$$\frac{\zeta + f}{H} = \frac{1}{H} \left[ f + \frac{\partial}{\partial x} \left( \frac{1}{H} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{H} \frac{\partial \psi}{\partial y} \right) \right] = \zeta,$$

A1

where ζ is a specified function.

Taking the divergence of the momentum equations and using the property that the flow is nondivergent, one can easily see that
sides. Only part of the computational domain is shown in Fig. 1. The bottom topography is given by function

\[ H(x, y) = 1.0 - 0.9 \exp \left[ -\frac{(x-x_0)^2 + (y-y_0)^2}{2r_0^2} \right] \]

where \( x_0 = 0.4, y_0 = 0.65 \) and \( r_0 = 0.12 \). The initial distribution of the potential vorticity is

\[ \tilde{\Psi}(x, y) = \hat{f} + \hat{\psi}_0 \]

where \( \hat{f} = 10 \) is the Coriolis parameter, \( \hat{\psi}_0 = 60 \) is the amplitude of the potential vorticity anomaly, \( \delta_x = 0.4, \delta_y = 0.30, \delta_z = 0.05, \) offset \( \delta = 0.03535 \). This distribution of the potential vorticity field induces velocity field with maximum value of order of 1, which is comparable to the speed of the self-advection of the dipole. The nondimensional acceleration of gravity in the experiments presented in Fig. 1 is \( g = 100 \), which corresponds to the nondimensional phase speed of the external gravity waves 10. We have also performed experiments with \( g = 50 \) and with rigid-lid approximation (formal limit of \( g \to \infty \)). Time step \( \Delta t = 0.002 \). Viscous dissipation coefficient was kept the same in all experiments, \( \alpha = 0.000075 \).

**APPENDIX B**

**The Accuracy of the Reflection Boundary Conditions**

Consider implementation of the no-slip boundary condition on a north–south wall. In this case, the coastal line goes through \( \hat{\nu} \) points and the \( \hat{\nu} \) component is set to zero, while the \( \hat{v} \) component is to be reflected. Suppose \( \hat{v}_{j,k} \) is the last \( \hat{v} \) point in the interior and \( \hat{v}_{j+1,k} \) belongs to the land. According to the reflection rule, the value of the \( \hat{v} \) component in the land point adjacent to the coastline is prescribed as \( \hat{v}_{j+1,k} = -\hat{v}_{j,k} \), so that linear interpolation of the velocity component to the coastline always yields zero. The standard finite-difference scheme is the approximation for the second derivative in the last interior point \( \hat{v}_{j,k} \) as follows:

\[
\frac{\partial^2 \hat{v}}{\partial x^2}
\]

\[
\bigg|_{j,k} = \hat{v}_{j+1,k} - 2\hat{v}_{j,k} + \hat{v}_{j-1,k}
\]

\[
\Delta x^2
\]

\[
= -3\hat{v}_{j,k} + \hat{v}_{j-1,k}
\]

\[ \left( \frac{\partial^2 H}{\partial x^2} \right) \]

\[ \left( \frac{\partial^2 \hat{H}}{\partial y^2} \right) \]

where Cartesian coordinates are used for simplicity.
Obviously,

\[ v_{j,k+1} = v_{j,k} + \frac{\partial v}{\partial x} \bigg|_{j,k} \frac{\Delta x}{2} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \bigg|_{j,k} \left( \frac{\Delta x}{2} \right)^2 + \frac{1}{6} \frac{\partial^3 v}{\partial x^3} \bigg|_{j,k} \left( \frac{\Delta x}{2} \right)^3 + \]

and

\[ \frac{\partial v}{\partial x} \bigg|_{j,k} \frac{\Delta x}{2} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \bigg|_{j,k} (\Delta x)^2 \]

\[ - \frac{1}{6} \frac{\partial^3 v}{\partial x^3} \bigg|_{j,k} (\Delta x)^3 + \]

and, therefore,

\[ 2v_{j+1/2,k} + v_{j-1/2,k} - 3v_{j,k} \]

\[ (\Delta x)^2 \]

Taking into account that \( v_{j+1/2,k} = 0 \) because this is the value of the \( v \) component on the boundary, one can easily see that expression (B1) approximates \((8/5)(\partial^2 v/\partial x^2)|_{j,k}\) rather than \( \partial^2 v/\partial x^2 |_{j,k} \).

REFERENCES


