4. Stability properties of various time differencing schemes.

4.1 Applied to the advection equation $\left(\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}\right)$

The FTCS is a one-step, explicit, two-time level method

One-step : One calculation step is required to advance to a new time level

Explicit : All the values on the right-side are known.

Two-time levels : Only two time levels are involved in the calculation.

a/ The Leap Frog scheme

The leap frog is centered in time which is unstable for the diffusive eqn and adv. diff eqn but when applied to the adv. eqn alone, is stable.

(1) $$U_j^{n+1} = U_j^{n-1} - \lambda\left(U_{j+1}^{n} - U_{j-1}^{n}\right)$$

$$U_j^{n} = \sum A_k^{n} e^{ikx_j}$$ k wave number $\frac{2\pi}{L}$

(2) $$\boxed{A_k^{n+1} = A_k^{n-1} - A_k^{n}\left(2i\lambda \sin(k\Delta x)\right)}$$

which can be rewritten in matrix form

$$\begin{pmatrix} A_k^{n+1} \\ A_k^{n} \end{pmatrix} = G\begin{pmatrix} A_k^{n} \\ A_k^{n-1} \end{pmatrix} \quad \text{with } G = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

$a = -2i\lambda\sin(k\Delta x)$

using the trivial $A_k^{n} = A_k^{n}$. G is now the

amplification factor. In this particular case of several time level, we use more conditions to fall back on a single two-time level. (See homework # 2)
Another way of presenting it is

$$\begin{cases} A_k^{n+1} = C_k^n - A_k^n \left(2i \lambda \sin(k \Delta x)\right) \\ C_k^{n+1} = A_k^n \end{cases}$$

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The stability criteria from Von Neuman is
$|\mu_i| \leq 1 + o(\Delta t)$ for all i where μ_i are the eigenvalue of the matrix G. Another sufficient condition slightly less restrictive is $\|G\| \leq 1 + o(\Delta t)$. Again see Richtmyer for a complete derivation.
Solving for the eigenvalues $\begin{vmatrix} (a-\mu) & 1 \\ 1 & (0-\mu) \end{vmatrix} = 0$

(3) $\boxed{\mu_{1,2} = \frac{1}{2}\left(a \pm \sqrt{a^2+4}\right)}$

or

(3') $\boxed{\mu_{1,2} = -i\lambda \sin(k\Delta x) \pm \sqrt{1 - \lambda^2 \sin^2(k\Delta x)}}$

* If $\lambda^2 \sin^2(k\Delta x) > 1$, the square root term is then imaginary and

$$\mu_{1,2} = +i\left(-\lambda \sin(k\Delta x) \pm \sqrt{\lambda^2 \sin^2(k\Delta x) - 1}\right)$$

$$|\mu_{1,2}|^2 = 2\lambda^2 \sin^2(k\Delta x) - 1 \pm 2\lambda \sin(k\Delta x)\sqrt{\lambda^2 \sin^2(k\Delta x) - 1}$$

$$= 2\lambda^2 \sin^2(k\Delta x) - 1 \pm 2\lambda^2 \sin^2(k\Delta x)\left[1 - \frac{1}{\lambda^2 \sin^2(k\Delta x)}\right]^{1/2}$$

obviously > 1

$\Rightarrow \boxed{\max |\mu_{1,2}| > 1}$

* If $\lambda^2 \sin^2(k\Delta x) < 1$ (True for $\lambda < 1$), then the module of $|\mu_i|$ is given by $|\mu_i|^2 = \lambda^2 \sin^2(k\Delta x) + (1 - \lambda^2 \sin^2(k\Delta x))] = 1$

This obviously satisfies the requirement for stability provided again that $\lambda < 1$ $\left(\frac{c\Delta t}{\Delta x} < 1\right)$

* Insert box 8

Any numerical method for the inviscid equation which exhibits an $|G|$ (or $S_R(G)$, $\|G\|$) < 1 exhibits an artificial damping. For any convergent method, the numerical damping error must, of course, vanish as $\Delta x, \Delta t \to 0$. In this particular case of the leap-frog, the damping is equal to zero for $c = c^t$ and $\lambda \leq 1$.

The leap-frog for $\lambda = 1$ perpetuates the exact solution for all time given an exact first time level solution. The inviscid equ. $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ is equivalent to say that

$$u(x, t+\tau) = u(x - c\tau, t)$$

If $\tau = \Delta t$, then for $c = 1$

(4) $$u_i^{n+1} = u_{i-1}^n$$

Over $2\Delta t$, the exact solution is

(5) $$u_i^{n+1} = u_{i-2}^{n-1}$$

Application of the leap frog method gives

(6) $$U_i^{n+1} = U_i^{n-1} - U_{i+1}^n + U_{i-1}^n$$

Given the correct starting values from (4)

$$U_{i+1}^n = U_i^{n-1} \quad \text{and} \quad U_{i-1}^n = U_{i-2}^{n-1}$$

then (6) is exactly equal to (5)

Two sets of values are required to start. If with an error then the error will persist in the calculation ...

The corresponding eigenvectors to $\mu_{1,2}$ are

$$\vec{x}_1 \begin{pmatrix} -i\lambda \sin k(\Delta x) + \sqrt{1-\lambda^2 \sin^2(k\Delta x)} \\ 1 \end{pmatrix}$$

$$\vec{x}_2 \begin{pmatrix} -i\lambda \sin(k\Delta x) - \sqrt{1-\lambda^2 \sin^2(k\Delta x)} \\ 1 \end{pmatrix}$$

$\vec{x}_1 \not\perp \vec{x}_2$, but they are independent, and any vector can be expressed as a function of $\vec{x}_1, \vec{x}_2$,

$$\begin{pmatrix} A_k(n+1) \\ A_k(n) \end{pmatrix} = B^n \begin{pmatrix} A_k(1) \\ A_k(0) \end{pmatrix}$$

$$= B^n \left(\alpha \vec{x}_1 + \beta \vec{x}_2 \right)$$

Then G_1, $\vec{x}_1$ are associated with the steady part of the solution (Physical mode)

G_2, $\vec{x}_2$ part of the solution that changes sign every time (Computational mode)

If we restrict a moment to the case where G' is equal for two consecutive time steps

$$G' = \frac{A_k(n+1)}{A_k(n)} = \frac{A_k(n)}{A_k(n-1)}$$

Illustrate the relationship between G and the eigenvalues μ_i

then $G'^2 + 2i\lambda \sin(k\Delta x) - 1 = 0$

and
$$\begin{cases} G'_1 = \mu_1 = -i\lambda \sin(k\Delta x) + \sqrt{1-\lambda^2 \sin^2 k\Delta x} \\ G'_2 = \mu_2 = \quad " \quad - \quad " \end{cases}$$

As $\Delta t \to 0$, $\mu_1 \to 1$ and $\mu_2 \to -1$.

Its origin lies in the fact that the solution of the FD are independent between odd and even numbers. These 2 solutions will evolve differently unless (a) the first time step generating $A_k(1)$ is executed such that $|\beta| \ll |\alpha|$ and (b) any component of the solution // to $\vec{x_2}$ that might ~~arise~~ arise due to round-off errors is periodically removed.

Uten

a)

$$A_k(0)\left[1 - ip\right] = \alpha\left[-ip + \sqrt{1-p^2}\right] + \beta\left[-ip - \sqrt{1-p^2}\right]$$

$$A_k(0) = \alpha + \beta$$

or

$$A_k(0)\left(1 - ip\right) = A_k(0)\left[-ip + \sqrt{1-p^2}\right] + \beta\left[-2\sqrt{1-p^2}\right]$$

$$\beta_1 = A_k(0)\frac{1 - \sqrt{1-p^2}}{-2\sqrt{1-p^2}} = A_k(0)\left[\frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{1-p^2}}\right]$$

b)

$$A_k(0) = \alpha\left[-ip + \sqrt{1-p^2}\right] + \beta\left[-ip - \sqrt{1-p^2}\right]$$

$$A_k(0) = \alpha + \beta$$

$$\beta_2 = A_k(0)\frac{1 + ip - \sqrt{1-p^2}}{-2\sqrt{1-p^2}} = A_k(0)\left[\frac{1}{2} - \frac{1+ip}{2\sqrt{1-p^2}}\right]$$

The two eigenvectors of the leap-frog scheme are

$$\vec{x_1} = \begin{pmatrix} -ip + \sqrt{1-p^2} \\ 1 \end{pmatrix} \qquad \vec{x_2} = \begin{pmatrix} -ip - \sqrt{1-p^2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} A_k(1) \\ A_k(0) \end{pmatrix} = \alpha \vec{x_1} + \beta \vec{x_2} \qquad p = \lambda \sin(k\pi\Delta x)$$

We are looking for the method which produces the smaller computational mode (smallest β) from starting

a) $U_j^1 = U_j^0 - \lambda/2 \, (U_{j+1}^0 - U_{j-1}^0)$

b) $U_j^1 = U_j^0$

or in function of the test function

a) $A_k(1) = A_k(0) - \lambda/2 \, A_k(0) (2i \sin(k\pi\Delta x$

b) $A_k(1) = A_k(0)$

In order to have a stable scheme, $\lambda \leq 1$, then $1-p^2 \geq 0$

and we can rewrite

$$\frac{|\beta_2|^2}{|\beta_1|^2} = \frac{\left(1-\sqrt{1-p^2}\right)^2 + p^2}{\left(1-\sqrt{1-p^2}\right)^2} \geq 1 \quad \text{everywhere.}$$

Then a leapfrog integration of the advection equation

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

should be started with a single forward step,

i.e $\quad U_j^1 = U_j^0 - \lambda/2 \left(U_{j+1}^0 - U_{j-1}^0 \right)$

(Smallest computational mode)

b)- Upstream differencing (Donor cell)

$$(7) \quad U_j^{n+1} = U_j^n - \lambda \begin{cases} U_{j+1}^n - U_j^n & \text{if } \lambda < 0 \\ U_j^n - U_{j-1}^n & \text{if } \lambda > 0 \end{cases}$$

The amplification factor G is then equal to

$$1 - \lambda \begin{cases} e^{ik\Delta x} - 1 & \text{if } \lambda < 0 \\ 1 - e^{-ik\Delta x} & \text{if } \lambda > 0. \end{cases}$$

This scheme is stable if $|\lambda| < 1$. Easy to implement, but not recommended because introduce artificial dissipation (computational viscosity).

(7) can be rewritten as

$$U_j^{n+1} = \underbrace{U_j^n - \lambda/2\left(U_{j+1}^n - U_{j-1}^n\right)}_{\substack{\text{FTCS} \\ \text{(unstable)}}}$$

$$+ \frac{|\lambda|}{2}\left(\underbrace{U_{j+1}^n + U_{j-1}^n - 2U_j^n}_{\text{second order approximation to a diffusion equ.}}\right]$$

$\alpha = \frac{c\Delta x}{2}$ represent the value for the eddy viscosity!

c) Diffusion scheme (also called Friedrich's scheme)
Euler-diffusion "
Forward- " "

Already presented before

$$U_j^{n+1} = 1/2\left(U_{j+1}^n + U_{j-1}^n\right) - \lambda/2\left(U_{j+1}^n - U_{j-1}^n\right)$$

The amplification factor is $G = \cos(k\Delta x) - i\lambda \sin(k\Delta x)$

Stable if $|\lambda| < 1$

this scheme also introduces diffusivity to stabilize the FTCS scheme.

$$\underbrace{U_j^{n+1} = U_n^j - \lambda/2\left(U_{j+1}^n - U_{j-1}^n\right)}_{FTCS} + \underbrace{1/2\left[U_{j+1}^n + U_{j-1}^n - 2U_j^n\right]}_{\substack{\text{diffusion} \\ \text{with } \alpha' = \frac{\Delta x^2}{2\Delta t} \\ \text{Artificial viscosity}}}$$

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d) - The Lax - Wendroff scheme

We saw that the previous two schemes did introduce artificial viscosity. The reasoning behind the Lax - Wendroff scheme is the following: Can we stabilize the FTCS scheme by adding the minimal amount of artificial damping? We can write the FD equation as

$$U_j^{n+1} = U_n^j - \lambda/2\left(U_{j+1}^n - U_{j-1}^n\right) + \nu\left(U_{j-1}^n + U_{j+1}^n - 2U_j^n\right)$$

$\nu = 0 \Rightarrow$ FTCS

$\nu = 1/2 \Rightarrow$ Diffusion scheme

$\nu = \frac{|\lambda|}{2} \Rightarrow$ Upstream scheme. (better than diffusion since $|\lambda| < 1$)

The amplification factor for such a scheme is

$$(8) \quad \boxed{G = \frac{A_k^{n+1}}{A_k^n} = 1 - i\lambda \sin(k\Delta x) + 2\nu\left(\cos(k\Delta x) - 1\right)}$$

Stability is assured if $|G| \leq 1 + O(\Delta t)$ for all k

$$(9) \quad |G|^2 = 1 + (2\lambda^2 - 4\nu)\left[1 - \cos(k\Delta x)\right] + \left(4\nu^2 - \lambda^2\right)\left[1 - \cos(k\Delta x)\right]^2$$

$$= 1 + (2\lambda^2 - 4\nu)\,\rho + (4\nu^2 - \lambda^2)\,\rho^2$$

For $\nu = \frac{|\lambda|}{2}$, G is a linear function of $p = 1 - \cos k\Delta x$

Scheme stable, upstream diff. scheme.

We look for a better scheme namely

$$\nu < \frac{|\lambda|}{2} \quad \text{or} \quad \lambda^2 > 4\nu^2$$

(9) is maximum for $\boxed{p_{max} = \frac{\lambda^2 - 2\nu}{\lambda^2 - 4\nu^2}}$

and $|G|^2$ is then equal to at this point

$$\boxed{1 + \frac{(\lambda^2 - 2\nu)^2}{\lambda^2 - 4\nu^2}}$$

Then

(10) $$\boxed{\max_{0 \le p \le 2} |G|^2 \le 1 + \frac{(\lambda^2 - 2\nu)^2}{\lambda^2 - 4\nu^2}}$$

(Maximum not necessarily located in $[0, 2]$)

This implies $(\lambda^2 - 2\nu)^2 = 0$ or $\boxed{\nu = \frac{\lambda^2}{2}}$

* Further reduction is not possible $\nu < \lambda^2/2 \Rightarrow$

(11) $$\max_{0 \le p \le 2} |G|^2 = \begin{cases} 1 + \dfrac{(\lambda^2 - 2\nu)^2}{\lambda^2 - 4\nu^2} > 1 & \text{for } p_{max} < 2 \\ \underline{(1 - 4\nu)^2} & \text{for } p_{max} > 2 \\ > 1 & \end{cases}$$

$$\left(\begin{array}{l} \nu > 1/2 \\ \lambda > 1 \end{array} \right.$$

$$p_{max} = \frac{\lambda^2 - 2\nu}{\lambda^2 - 4\nu^2} > 2$$

$$\Rightarrow \lambda^2 - 2\nu > 2\lambda^2 - 8\nu^2$$

$$\lambda^2 - 8\nu^2 + 2\nu < 0$$

$$\left\{ \begin{array}{l} \lambda^2 < 8\nu^2 - 2\nu \\ \lambda^2 > 4\nu^2 \end{array} \right.$$

$$4\nu^2 < 8\nu^2 - 2\nu$$

$$0 < 4\nu^2 - 2\nu$$

$$\nu(4\nu - 2)$$

$$\Rightarrow \left\{ \begin{array}{l} \nu > 1/2 \\ \lambda > 1 \end{array} \right.$$

$$\nu < \lambda^2/2 \Rightarrow \nu > 1/2$$

This leads to the Lax-Wendroff scheme

(12) $$\boxed{U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\lambda^2}{2}\left(U_{j-1}^n + U_{j+1}^n - 2U_j^n\right)}$$

Artificial viscosity $\alpha = \Delta t/2$

This scheme is <u>second order</u> accuracy in time (and space) because of the cancellation of term Δt^2 due to $D = \lambda^2/2$.

This scheme can also be obtained from a Taylor series expansion in time

$$U_j^{n+1} = U_j^n + \Delta t \frac{\partial u}{\partial t}\Big|_j^n + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}\Big|_j^n + \dots$$

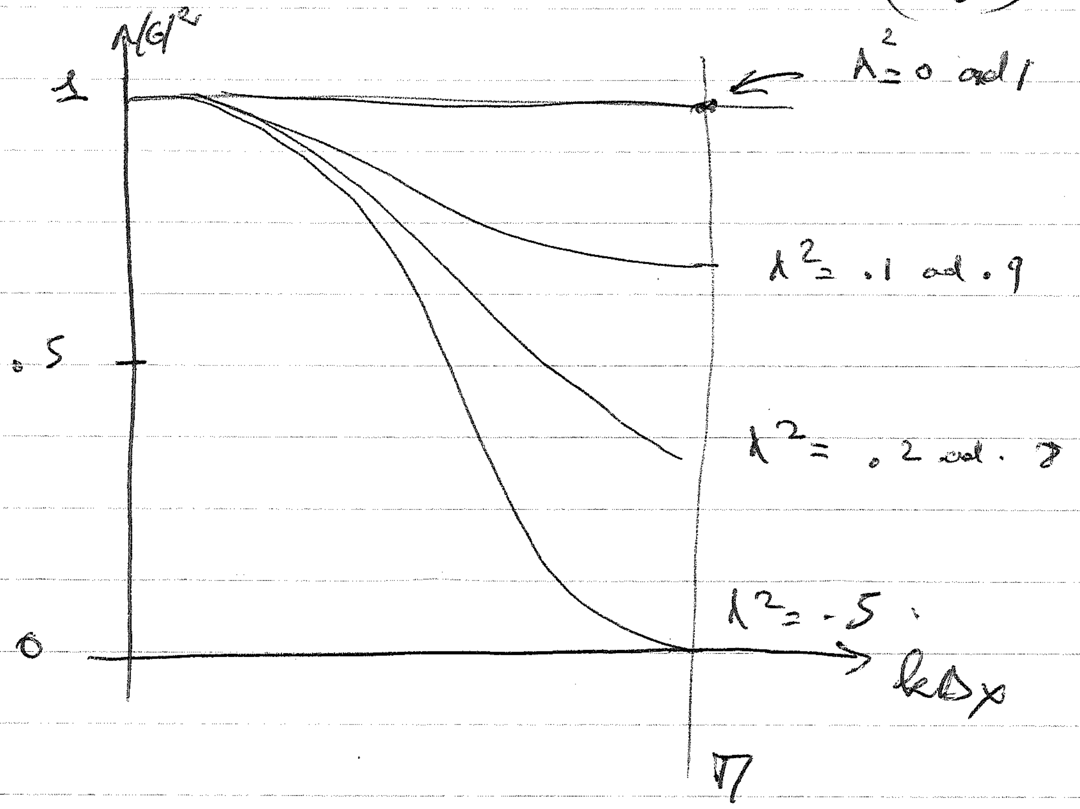
$$= U_j^n - c\,\Delta t \frac{\partial u}{\partial x}\Big|_j^n + \frac{\Delta t^2 c^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_j^n + O(\Delta t^3)$$

$$\left(\frac{\partial^2 u}{\partial t^2} = -c\frac{\partial}{\partial x\,\partial t}u = c^2\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow\quad U_j^{n+1} = U_j^n - \frac{\overbrace{c\,\Delta t}^{\lambda}}{2\Delta x}\left[U_{j+1}^n - U_{j-1}^n\right] + \frac{c^2\Delta t^2}{2\Delta x^2}\left[U_{j+1}^n + U_{j-1}^n - 2U_j^n\right] + O(\Delta t^3)$$

The amplification factor $|G|^2$ for such a scheme is then

$$|G|^2 = 1 - 4(\lambda^2 - \lambda^4)\sin^4\left(\frac{k\Delta x}{2}\right)$$



Except for $\lambda = 0, 1$, there is a damping effect

Avoid a time derivative

Two-step method is better suited for 2D pbs and non linear problems. ~~Half Odd~~ time step are not considered a solution of the FD eqn.

Step 1 Diffusing time step

$$U_{j+1/2}^{n+1/2} = \frac{1}{2}\underbrace{\left(U_{j+1}^{n} + U_{j}^{n}\right)}_{U_{j+1/2}^{n}} - \frac{c\Delta t}{2\Delta x}\underbrace{\left(U_{j+1}^{n} - U_{j}^{n}\right)}_{\left.\frac{\partial u}{\partial x}\right|_{j+1/2}}$$

$f_{n+1/2}$

Step 2 Leap Frog

$$U_{j}^{n+1} = U_{j}^{n} - \lambda\left(U_{j+1/2}^{n+1/2} - U_{j-1/2}^{n+1/2}\right)$$

It is important to retain the spatial staggering of the values otherwise excessive diffusion

Feb 14/94

e.) **The Euler-Backward (Matsuno) scheme**

Predictor-Corrector scheme

If we used an explicit scheme to achieve a guess for U_k^{n+1} and an implicit scheme for the final result, where the explicit guess is used on the right ~~side~~ hand side to make the second step explicit. This two-time step is called a Predictor-Corrector scheme

Step 1 FTCS

(13) $$\boxed{\overset{*}{U}{}_{j}^{n+1} = U_{j}^{n} - \frac{\lambda}{2}\left(U_{j+1}^{n} - U_{j-1}^{n}\right)}$$

Step 2 Backward step

$$U_{j}^{n} = U_{j}^{n+1} + \frac{\lambda}{2}\left(\overset{*}{U}{}_{j+1}^{n+1} - \overset{*}{U}{}_{j-1}^{n+1}\right) \quad \text{or}$$

(14) $$\boxed{U_{j}^{n+1} = U_{j}^{n} - \frac{\lambda}{2}\left(\overset{*}{U}{}_{j+1}^{n+1} - \overset{*}{U}{}_{j-1}^{n+1}\right)}$$

If we eliminate all the asterisk values, then

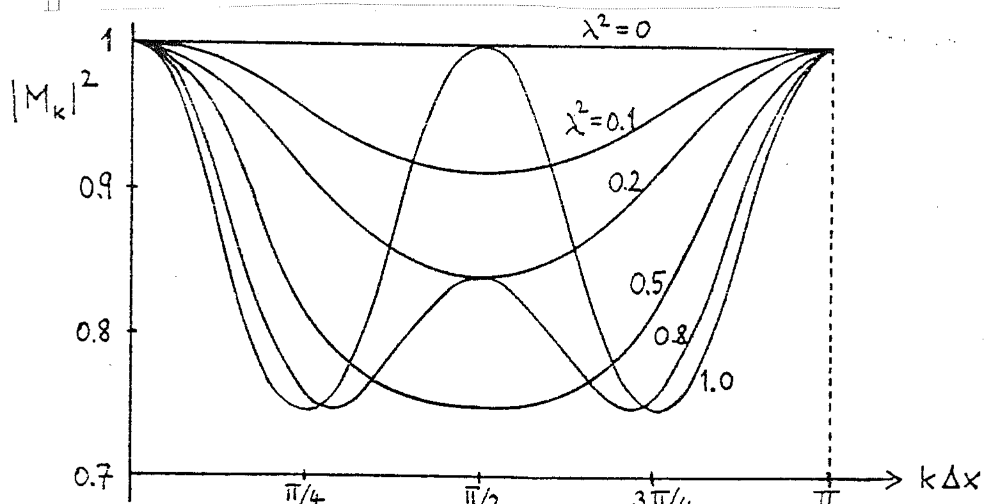
(15) $$U_j^{n+1} = U_j^n - \lambda/2\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\lambda^2}{4}\left(U_{j-2}^n + U_{j+2}^n - 2U_j^n\right)$$

Add. also an artificial damping of $\alpha^2 = c\,\Delta t$ twice as large as the Lax-Wendroff scheme. Qualitatively, long waves are more damped than in the Lax-Wendroff scheme. On the other hand, the shortest wave $L = 2\Delta x$ is not damped at all contrary to LW scheme. Accuracy is only of the first order in time.

The absolute value of the amplification factor $|G|$ is given by

(16) $$|G|^2 = 1 - \lambda^2 \sin^2(k\Delta x) + \lambda^4 \sin^4(k\Delta x)$$

Stable as long as $|\lambda| < 1$



All curves are symmetric for $k\Delta x = \pi/2$.
In the wavelength interval $8\Delta x \le L \le \infty$ ($k\Delta x \le \pi/4$) the damping becomes stronger the faster the waves moves. In this regard, Rostrsseno is more desirable than LW were the trend is reversed at $\lambda^2 \doteq .5$ However waves shorter than $L = 8\Delta x$ are not selectively

damped according to their phase speed. Fast moving $2\Delta x$ waves are not damped at all whereas $3\Delta x$ waves ($k\Delta x = 2\pi/3$) are strongly damped

Application: Linearized shallow water equs

$$\begin{cases} \dfrac{\partial u}{\partial t} + g\dfrac{\partial h}{\partial x} = 0 \\ \dfrac{\partial h}{\partial t} + H_0 \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

Step 1

$$\begin{cases} U_j^{*\,n+1} = U_j^n - \dfrac{g\Delta t}{2\Delta x}\left(H_{j+1}^n - H_{j-1}^n\right) \\ H_j^{*\,n+1} = H_j^n - \dfrac{H_0\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) \end{cases}$$

Step 2

$$\begin{cases} U_j^{n+1} = U_j^n - \dfrac{g\Delta t}{2\Delta x}\left(H_{j+1}^{*\,n+1} - H_{j-1}^{*\,n+1}\right) \\ H_j^{n+1} = H_j^n - \dfrac{H_0\Delta t}{2\Delta x}\left(U_{j+1}^{*\,n+1} - U_{j-1}^{*\,n+1}\right) \end{cases}$$

or

$$\begin{cases} U_j^{n+1} = U_j^n - \dfrac{g\Delta t}{2\Delta x}\left(H_{j+1}^n - H_{j-1}^n\right) + gH_0\dfrac{\Delta t^2}{4\Delta x^2}\left(U_{j-2}^n + U_{j+2}^n - 2U_j^n\right) \\ H_j^{n+1} = H_j^n - \dfrac{H_0\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) + gH_0\dfrac{\Delta t^2}{4\Delta x^2}\left(H_{j-2}^n + H_{j+2}^n - 2H_j^n\right) \end{cases}$$

The result is then a scheme which damps both U and H at a coefficient $\alpha = gH_0\Delta t$ proportional to $c^2 = gH_0$ (surface gravity wave speed). It provides a method to selectively remove gravity waves from the solution without affecting the Rossby waves (slow moving). An explicit diffusion term can dampen waves only according to their spatial characteristics, not according to their speeds.

f) The Adam-Bashforth scheme

When comparing the Euler (FTCS) unstable scheme with the Lax Wendroff and Leap-Frog, it seems that centering the scheme in time greatly improve the numerical stability.

The Adam-Bashforth scheme does it by evaluating the RHS of the FD equ at two time level $n-1$, n and extrapolated to $n+1/2$.

linear extrapolation (17) Why 3/4 and 1/4?

$$\boxed{U_j^{n+1} = U_j^n - \tfrac{3}{4}\lambda\left(U_{j+1}^n - U_{j-1}^n\right) + \tfrac{\lambda}{4}\left(U_{j+1}^{n-1} - U_{j-1}^{n-1}\right)}.$$

The amplification matrix G is $\quad n \quad n+1/2 \quad n-1$

(derived as for leap frog)

$$G = \begin{pmatrix} 1 - \tfrac{3}{2} i\lambda \sin(k\Delta x) & \tfrac{i\lambda}{2}\sin k\Delta x \\ 1 & 0 \end{pmatrix}$$

The eigenvalues are given by

$$\mu_{1,2} = \tfrac{1}{2}\left(\left(1 - \tfrac{3}{2} i\lambda\sin(k\Delta x)\right) \pm \sqrt{1 - \tfrac{9}{4}\lambda^2 \sin^2(k\Delta x) - i\lambda\sin(k\Delta x)}\right)$$

$$= \tfrac{1}{2}\left(\left(1 - \tfrac{3}{2} i\lambda \sin k\Delta x \pm \sqrt{1 - \tfrac{9}{4}\lambda^2 \sin^2 k\Delta x - i\lambda \sin k\Delta x}\right)\right)$$

As $\sin(kx) \to 0$, we note that $\mu_1 \to 1$

$\mu_2 \to 0$

Thus μ_1 is associated with the "physical" mode of the solution which progresses in an orderly fashion whereas the "computational" mode associated with μ_2 is being damped. This is an important aspect (advantage) of the AB scheme.

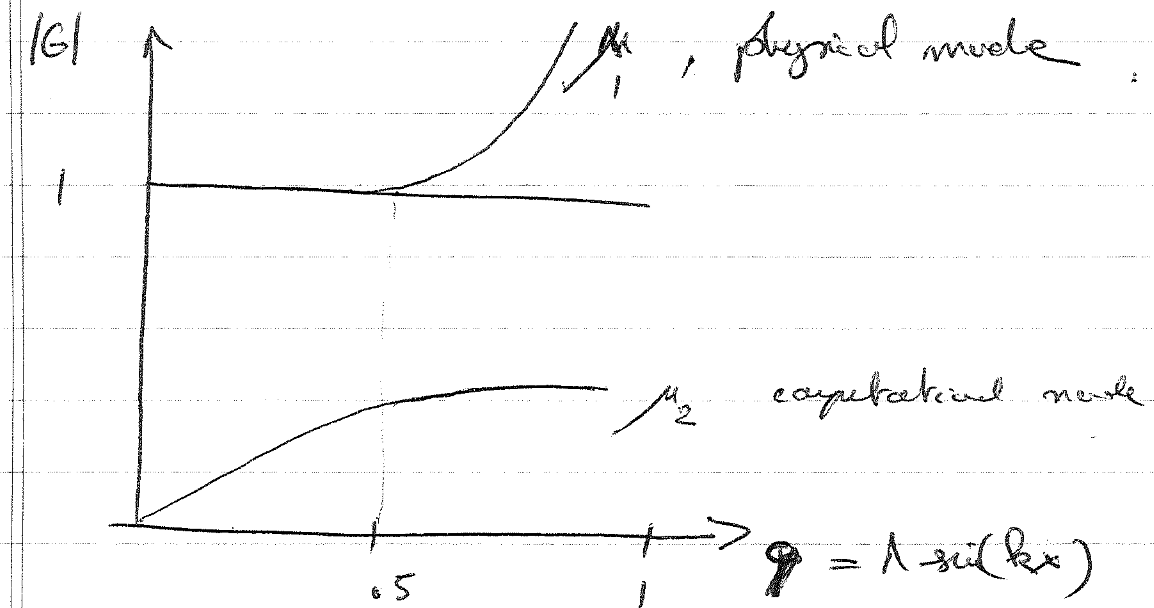
What is the magnitude of μ_1?

If $q = \lambda \sin(k\Delta x)$, then

$$\mu_1 = 1 - iq - \frac{q^2}{2} - \frac{iq^3}{4} - \frac{q^4}{8} + \cdots$$

$$|\mu_1| = 1 + \frac{p^4}{4} + \cdots$$

This result shows that the physical mode of the solution is actually slightly unstable, but $|\mu_1| = 1 + O(\Delta t^4)$, then if Δt small enough and with small dissipation



Starts to deviate for $q > .5$. So a realistic limit is for $|\lambda| = .5$. Unstable growth if any will be first noticeable at $L = 4\Delta x$

g) Transport schemes.

Transport schemes have been specifically developed for modeling advective processes and are improvements over the "classic" schemes.
For a reference list, see the review paper by
Rood R. B., 1987, Reviews of Geophysics vol 25, pages 71 – 100.

Two classics : Flux Corrected Transport; Smolarkiewicz scheme
(See homework # 5)

h) Implicit schemes

Implicit schemes use advance values (at n+1) to advance the calculation at n+1. We therefore need the simultaneous solution of equations (n+1) in order to advance the calculation. They therefore require a higher level of mathematical sophistication than explicit schemes (inversion of matrices), but they are attractive because of their stability properties.

Explicit schemes are simple and cost per time step low. The sources of the errors are easily identified and therefore can be alleviated or removed. It is sometimes necessary to take a very small time step to have the scheme stable.

Implicit are stable for any time step. The schemes are either iterative or require inversion of big matrices ⇒ costly per time step. These schemes tend to have dispersion errors by retardation of short wave features.

$$u^{n+1} = u^{n} - c\Delta t \frac{\partial u}{\partial x}$$ for the inviscid adv. eqn.

* If we define $\frac{\partial u}{\partial x}$ at $n+1$ using centered in space then

(18) $$U_j^{n+1} = U_j^{n} - \frac{c\Delta t}{2\Delta x}\left(U_{j+1}^{n+1} - U_{j-1}^{n+1}\right)$$

also called the "<u>backward scheme</u>"
([illegible])

The amplification factor for such a scheme is

$$G = \frac{1}{1 + i\lambda \sin(k\Delta x)}$$

and

$$|G|^2 = \frac{1}{1 + \lambda^2 \sin^2(k\Delta x)}$$

We therefore have $|G| \leq 1$ regardless of λ with a slight damping effect on the solution. The scheme is <u>unconditionally</u> stable. This is the major advantage of implicit over explicit.

* Another implicit scheme is the "<u>trapezoidal</u>" scheme
Crank

(19) $$U_j^{n+1} = U_j^{n} - \lambda/2\left(\frac{U_{j+1}^{n} + U_{j+1}^{n+1}}{2} - \frac{U_{j-1}^{n} + U_{j-1}^{n+1}}{2}\right)$$

$$G = \frac{1 - \frac{i\lambda}{2}\sin k\Delta x}{1 + \frac{i\lambda}{2}\sin k\Delta x}$$

$$|G|^2 = 1$$ exactly. (also called sometimes Euler's modified method)

(Euler = FCTS)

Actually, the explicit Euler-Backward scheme
(Matsuno)

was designed with the stable "backward" scheme in mind. One can try the same idea with the "trapezoidal" scheme. This leads to the "Heun" scheme slightly unstable as in the Adams-Bashforth scheme. Can be reduced with small Δt and diffusion.

(20) $$U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\lambda^2}{8}\left(U_{j+2}^n + U_{j-2}^n - 2U_j^n\right)$$

Feb 22, 93 "Heun" scheme.

() – Phase Errors.

Numerical stability is clearly the overriding criterion for the usefulness of a finite difference scheme, one should also pay attention the differences in accuracy of the schemes.
The truncation error is a useful measure but is more a representation of the gain that can be obtained by mesh refinement as opposed to the actual magnitude of the error in a given mesh. Another measure which complement the Fourier analysis of Chapter 3 is the phase error of the individual Fourier modes.

To determine this error we look at the phase shift per time step (angular change) of each component of the exact solution to the phase shift of the respective Fourier component of the numerical solution.

In the exact solution, all Fourier component experience a phase shift proportional to k, namely

$-kc\Delta t$. If the error is zero, then the scheme is ~~termed~~ as "non-dispersive"

The phase shift of the components of the numerical solution is given by the angle of the complex number G and the real axis. The relative phase error is the relative deviation between the phase shift of the exact solution and of the numerical solution

$$R_\theta = \frac{\operatorname{atan}\frac{Im(G)}{Re(G)} - (-kc\Delta t)}{-kc\Delta t}$$

(21)
$$R_\theta = -\frac{1}{kc\Delta t}\underbrace{\operatorname{atan}\frac{Im(G)}{Re(G)}}_{\theta} - 1$$

Let's distinguish between the phase errors induced by the spatial differencing and those by the time differencing.

<u>spatial differencing</u>.

$$\frac{\partial U_j}{\partial t} = -\frac{c}{2\Delta x}\left(U_{j+1} - U_{j-1}\right)$$

$$U_j(t) = \sum B_k(t)\, e^{ikx_j}.$$

$$\Rightarrow \frac{\partial}{\partial t} B_k(t) = -\frac{ic}{\Delta x}\sin k\Delta x \, B_k(t)$$

$$= -ikc \boxed{\frac{\sin(k\Delta x)}{k\Delta x}} B_k(t)$$

Then the spatial differencing reduces the advection velocity c by the factor $\dfrac{\sin(k\Delta x)}{k\Delta x}$.

Now if we want to isolate the phase error induced by the time differencing scheme, we can replace $\frac{\sin k\Delta x}{k\Delta x}$ by 1 or $\sin(k\Delta x) = k\Delta x$ (True for small values).

<u>Leap-Frog</u>

Eigenvalue associated with the physical mode is

$$\mu_1 = -i\lambda k\Delta x + \sqrt{1-\lambda^2 k^2 \Delta x^2}$$

$$R_\theta = -\frac{1}{\lambda k \Delta x} \operatorname{atan} \frac{-\lambda k \Delta x}{\sqrt{1-\lambda^2 k^2 \Delta x^2}} - 1$$

$$= -\frac{1}{\lambda k \Delta x} \operatorname{atan}\left(\lambda k \Delta x + \frac{(\lambda k \Delta x)^3}{2} + \dots\right) - 1$$

$$= \frac{\lambda^2}{6} k^2 \Delta x^2 + \dots$$

The leap frog scheme has an accelerating effect which is compensated by the retarding effect of the spatial differencing.

* Let's now combine the two

Phase shift during one time step θ

Leap-frog:

$$\mu_1 = -i\lambda \sin(k\Delta x) + \sqrt{1-\lambda^2 \sin^2(k\Delta x)}$$

$$\tan\theta = \frac{-\lambda \sin(k\Delta x)}{\sqrt{1-\lambda^2 \sin^2(k\Delta x)}}$$

$$\begin{cases} \sin x = x - x^3/6 + x^5/5! + \cdots \\ \operatorname{atan} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} \\ (1+x)^\alpha = 1 + \dfrac{\alpha}{1!} x + \dfrac{\alpha(\alpha-1)}{2!} x^2 + \cdots \end{cases}$$

$$R_\theta = -\Big(\underbrace{1/6}_{\text{spatial}} - \underbrace{\frac{\lambda^2}{24}}_{\text{time diff.}}\Big) k^2 \Delta x^2 + o(k^4 \Delta x^4)$$

Upstream - Differencing

$$\tan\theta = \frac{-\lambda \sin(k\Delta x)}{1 - |\lambda|(1-\cos(k\Delta x))}$$

$$R_\theta = -\left(1/6 - \frac{|\lambda|}{2} + \frac{\lambda^2}{3}\right) k^2 \Delta x^2 + o(k^4 \Delta x^4)$$

Lax - Wendroff.

$$\tan\theta = \frac{-\lambda \sin(k\Delta x)}{[1-\lambda^2(1-\cos k\Delta x))}$$

$$R_\theta = -\left(1/6 - \lambda^2/6\right) k^2 \Delta x^2 + o(k^4 \Delta x^4)$$

Matsuno

$$\tan\theta = \frac{-\lambda \sin(k\Delta x)}{1-\lambda^2 \sin^2(k\Delta x)}$$

$$R_\theta = -\left(\tfrac{1}{6} - \frac{2\lambda^2}{3}\right) k^2\Delta x^2 + O(k^4\Delta x^4)$$

Heun

$$\tan\theta = \frac{-\lambda \sin(k\Delta x)}{1-\tfrac{1}{2}\lambda^2 \sin^2 k\Delta x}$$

$$R_\theta = -\left(\tfrac{1}{6} - \tfrac{\lambda^2}{6}\right) k^2\Delta x^2 + O(k^4\Delta x^4)$$

Backward s.

$$\tan\theta = \mp\lambda \sin(k\Delta x)$$

$$R_\theta = -\left(\tfrac{1}{6} + \frac{\lambda^2}{3}\right) k^2\Delta x^2 + O(k^4\Delta x^4)$$

more dispersive than the others

Trapezoidal

$$\tan\theta = \mp\frac{\lambda \sin(k\Delta x)}{1-\frac{\lambda^2}{4}\sin^2(k\Delta x)}$$

$$R_\theta = -\left(\tfrac{1}{6} + \frac{\lambda^2}{12}\right) k^2\Delta x^2 + O(k^4\Delta x^4)$$

FCT

$$\tan\theta = \frac{-\lambda \sin(k\Delta x)}{\left(1-\left(\frac{1}{4}+\lambda^2\right)\right)\left(1-\cos(k\Delta x)\right)}$$

$$R_\theta = -\left(\frac{1}{24} - \frac{\lambda^2}{6}\right) k^2 \Delta x^2 + O(k^4 \Delta x^4)$$

All algorithms have errors in $k^2\Delta x^2$.
For most problems, the λ^2 can be neglected $\lambda^2 \ll 1$.
The Forward Diffusive can reduce the $1/6$ term when ϵ is large. Otherwise FCT is smaller than the others. Implicit schemes add to the dispersive error.

Summary →

4.2 Applied to the diffusion equation

4.2.1 Laplacian friction $\left(\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \right)$ $= (\nabla^2 u)$

a) Euler scheme (FTCS)

$$U_j^{n+1} = U_j^n + \mu \underbrace{\left(U_{j-1}^n + U_{j+1}^n - 2U_j^n \right)}_{= \delta_2 U_j^n} \qquad \text{with } \mu = \frac{\alpha \Delta t}{\Delta x^2}$$

$$G = 1 + 2\mu(\cos k\Delta x - 1) \qquad \mu \geq 0$$

$$= 1 - 4\mu \sin^2 \frac{k\Delta x}{2}$$

Stable if $\mu \leq 1/2$.

(See homework #1)

b) Leap Frog (CTCS)

$$U_j^{n+1} = U_j^{n-1} + 2\mu\, \delta_2 U_j$$

$$\begin{pmatrix} A_k^{n+1} \\ C_k^{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} -8\mu \sin^2 \frac{k\Delta x}{2} & 1 \\ 1 & 0 \end{pmatrix}}_{= G} \begin{pmatrix} A_k^n \\ C_k^n \end{pmatrix}$$

$$\mu_{1,2} = -4\mu \sin^2 \frac{k\Delta x}{2} \pm \sqrt{1 + \left(4\mu \sin^2 \frac{k\Delta x}{2}\right)^2}$$

The computational mode is unconditionally unstable.

$\Rightarrow$ unstable scheme.

c) Adams - Bashforth

$$U_j^{n+1} = U_j^n - 3/4\, \mu\, \delta_2 U_j^n + \mu/4\, \delta_2 U_j^{n-1}$$

$$G = \begin{pmatrix} 1 - 6\mu \sin^2 \frac{k\Delta x}{2} & 2\mu \sin^2 \frac{k\Delta x}{2} \\ 1 & 0 \end{pmatrix}.$$

$$\mu_{1,2} = \frac{1}{2}\left(\left(1-6\mu \sin^2\frac{k\Delta x}{2}\right) \pm \sqrt{1-4\mu \sin^2\frac{k\Delta x}{2} + 36\mu^2 \sin^4\frac{k\Delta x}{2}}\right)$$

For $\mu \geq 0$, their variations can be determined by studying the function $y = \frac{1}{2}\left(1-3x \pm \sqrt{1-2x+9x^2}\right)$

Physical mode (+ sign) $y' < 0$ for all x $\quad y(0)=1$, $y(\infty)=0$

$\Longrightarrow$ physical mode stable for all values

Computational mode (− sign) $y' < 0$ $\quad y < -1$ for $x > \frac{1}{2}$

$\Longrightarrow$ stable for $\mu \leq 1/4$

d) Implicit Backward scheme

$$U_j^{n+1} = U_j^n + \mu\, \delta_2 U_j^{n+1}$$

$$G = \frac{1}{1+4\mu \sin^2\frac{k\Delta x}{2}}$$

Unconditionally stable

e) Implicit Trapezoidal scheme

$$U_j^{n+1} = U_j^n + \mu\left(\delta_2\left(\frac{U_j^n + U_j^{n+1}}{2}\right)\right)$$

$$G = \frac{1-2\mu \sin^2\frac{k\Delta x}{2}}{1+2\mu \sin^2\frac{k\Delta x}{2}}$$

Unconditionally stable

f) Implicit general case

$$U_j^{n+1} = U_j^n + \mu\left(a\, \delta_2 U_j^{n+1} + (1-a)\, \delta_2 U_j^n\right)$$

* If $0 < a < 1/2$, the scheme is stable if $\mu < (2-4a)^{-1}$
* If $1/2 \leq a \leq 1$, the scheme is always stable.

d) and e) are special cases. Another classic is the Crank-Nicholson scheme where $a = 1/2$.

g) <u>Euler - Backward scheme</u>

<u>2 Steps</u>: 1) $U_j^{*\,n+1} = U_j^n + \mu\, \delta_2 U_j^n$
2) $U^{n+1} = U_j^n + \delta_2 U_j^{*\,n+1}$

$$G = 1 - 4\mu \sin^2 \frac{k\Delta x}{2} + \left(4\mu \sin^2 \frac{k\Delta x}{2}\right)^2.$$

Stable if $\mu \leq 1/4$

h) <u>Du Fort - Frankel scheme</u>

Leap - Frog modified

$$U_j^{n+1} = U_j^{n-1} + 2\mu \left(U_{j-1}^n + U_{j+1}^n - 2\left(\frac{U_j^{n-1} + U_j^{n+1}}{2}\right)\right)$$

or

$$U_j^{n+1} = U_j^{n-1} + 2\mu \left(U_{j-1}^n + U_{j+1}^n - U_j^{n-1} - U_j^{n+1} \right)$$

$$G = \begin{pmatrix} \frac{4\mu \cos k\Delta x}{1+2\mu} & \frac{1-2\mu}{1+2\mu} \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of the matrix G are

$$\mu_{\frac{1}{2}} = \frac{2\mu \cos k\Delta x}{1+2\mu} \pm \sqrt{\frac{1-4\mu^2 \sin^2 k\Delta x}{(1+2\mu)^2}}.$$

Two cases:

1) $4\mu^2 \sin^2 k\Delta x > 1$

$$|\mu_{\frac{1}{2}}|^2 = \frac{4\mu^2 - 1}{(2\mu+1)^2} = \frac{2\mu-1}{2\mu+1} < 1 \quad \text{for all } \mu$$

2) $4\mu^2 \sin^2 k\Delta x < 1$

$$|\mu_{\frac{1}{2}}| \leq \frac{2\mu\,|\cos k\Delta x|}{1+2\mu} + \sqrt{\frac{1-4\mu^2\sin^2 k\Delta x}{(1+2\mu)^2}}$$

$$\leq \frac{1+2\mu\,|\cos k\Delta x|}{1+2\mu} \leq 1 \quad \text{for all } \mu$$

Unconditional stable. The eigenvalues become complex for $\mu > 1/2$ which creates a fictitious phase propagation which greatly reduces the accuracy of the solution. Very often used in oceanography.

4.2.2. Biharmonic friction $\left(\frac{\partial u}{\partial t} = -A_4 \nabla^4 u\right)$

* Laplacian Friction is the most common approximation for turbulence $\left(\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}\right)$
  Each component is damped $B_k(t) = B_0\, e^{-k^2 \alpha t}$
  The smallest waves $L_{min} = \frac{2\pi}{2\Delta x}$ are damped the fastest

* Often we cannot use very small grid spacing. How can we damp the smallest scales.
  Biharmonic friction $\left(\frac{\partial u}{\partial t} = \ominus A_4 \frac{\partial^4 u}{\partial x^4} = -A_4 \nabla^4 u\right)$ need for it
  Each component is damped $B_k(t) = B_0\, e^{-k^4 A_4 t}$.
  It damps the waves according to k^4 instead of k^2

Scale selective. The rate of damping decreases faster for longer waves.
Decay rates: $\delta_2 = 2\alpha (2\pi)^2/L^2$

$\delta_4 = 4 A_4 (2\pi)^4/L^4$

Example:

For grid scale waves, $L = 2\Delta x$, biharmonic friction more rapidly damps than Laplacian. $L = 4\Delta x$, the damping is equal, and for $L > 4\Delta x$, biharmonic damping drops rapidly. For $L = 20\Delta x$, for example, biharmonic damping is only 4 % of Laplacian. (After Holland, OPO, 1978) for $A_M = \alpha = 100\ m^2 s^{-1}$ and $A_4 = 8 \times 10^9\ m^4 s^{-1}$.

4.3 Combined Advective - Diffusive equations

The leap-frog is desirable because of its amplitude gain G is equal to 1. However it is unstable when combined with the diffusion equation. The simplest way to achieve a stable combination is leapfrog (CTCS) for advection and Euler (FTCS) for diffusion.

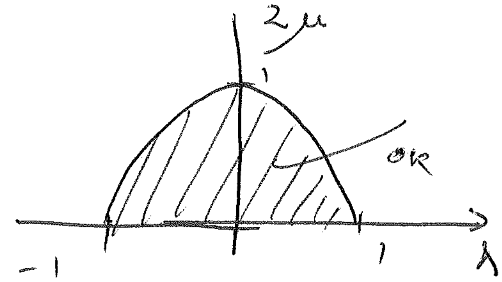
(22) $$\boxed{U_j^{n+1} = U_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(U_{j+1}^{n} - U_{j-1}^{n}\right) + \frac{2\alpha\Delta t}{(\Delta x)^2}\left[U_{j+1}^{n-1} + U_{j-1}^{n-1} - 2U_j^{n-1}\right]}$$

The scheme is stable if the amplification factor is less than 1.

<u>Necessary and sufficient conditions</u> (O'Brien, 1985)

$$\frac{c\,\Delta t^2 + 4\,\kappa\,\Delta t}{\Delta x^2} \leq 1$$

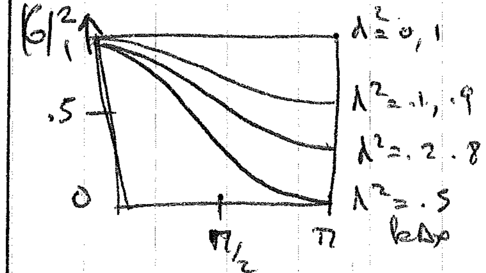
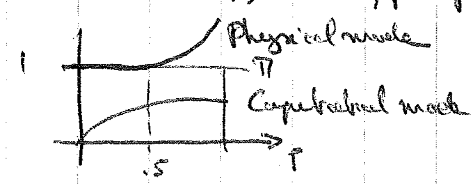
(or $\lambda^2 + 4\mu \leq 1$)

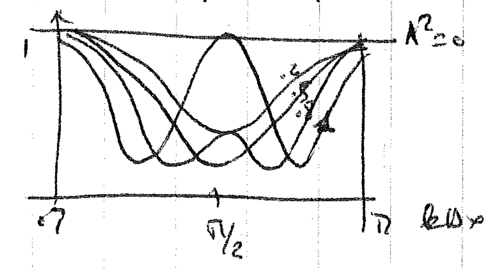


Another popular scheme is the use of Dufort - Frankel for the diffusion term

(23) $$\boxed{U_j^{n+1} = U_j^{n-1} - \lambda\left(U_{j+1}^{n} - U_{j-1}^{n}\right) + \mu\left(U_{j+1}^{n} + U_{j-1}^{n} - U_j^{n+1} - U_j^{n-1}\right)}$$

② Method	Finite-Difference Form	Amplification Factor / Stability	Truncation Error (order of accuracy)	Phase shift
Analytical	$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ (Advective equation) $\boxed{\lambda = c\frac{\Delta t}{\Delta x}}$ $\boxed{p = \lambda \sin(k\Delta x)}$	Represent the numerical solution as a Fourier series. Consider only one component. Numerical solution $f(A_k^{n+1}, A_k^n, \dots)$ $G = \frac{A_k^{n+1}}{A_k^n}$ if two-time level otherwise matrix $\begin{pmatrix} A_k^{n+1} \\ B_k^{n+1} \\ C_k^{n+1} \\ \vdots \end{pmatrix} = G \begin{pmatrix} A_k^n \\ B_k^n \\ C_k^n \\ \vdots \end{pmatrix}$ Von Neumann Condition for stability $\lvert G\rvert \leq 1 + O(\Delta t)$ if 2 time-level $S_k(G) = \max \lvert\mu_i\rvert \leq 1 + O(\Delta t)$ if matrix	Insert Taylor series expansions of the true solution into the FD equation. n = give order of accuracy	Exact solution: All Fourier component experience a phase shift of $-kc\Delta t$ Numerical solution: $\theta = \operatorname{atan}\frac{\operatorname{Im}(G)}{\operatorname{Re}(G)}$ $(\approx S_k(G))$ Relative phase error $R_\theta = -\frac{1}{\lambda k\Delta x}\theta - 1$
FTCS or (Euler)	$U_j^{n+1} = U_j^n - \lambda/2\,(U_{j+1}^n - U_{j-1}^n)$	$G = 1 - ip$ $\lvert G\rvert^2 = 1 + p^2$ Unstable	$O(1)$	$-\left(1/6 + \frac{\lambda^2}{3}\right)k^2\Delta x^2 + O(k^4\Delta x^4)$
Forward-Diffusive	$U_j^{n+1} = 1/2\,(U_{j+1}^n + U_{j-1}^n) - \lambda/2\,(U_{j+1}^n - U_{j-1}^n)$ $= U_j^n - \lambda/2\,(U_{j+1}^n - U_{j-1}^n) + 1/2\,(U_{j-1}^n + U_{j+1}^n - 2U_j^n)$ Computational viscosity $\alpha' = \frac{\Delta x^2}{2\Delta t}$	$G = \cos(k\Delta x) - ip = \sqrt{1 - p^2/\lambda^2} - ip$ $\lvert G\rvert^2 = 1 + p^2(1 - 1/\lambda^2)$ Stable if $\lambda \leq 1$	$O(2)$	$-\left(\cancel{1/6 + \frac{\lambda^2}{3} - \frac{1}{2\lambda^2}}\right)k^2\Delta x^2 + O(k^4\Delta x^4)$? $-(1/3 + \lambda^2/3)$
Leap Frog (CTCS)	$U_j^{n+1} = U_j^{n-1} - \lambda(U_{j+1}^n - U_{j-1}^n)$	$G = \begin{pmatrix} -2ip & 1 \\ 1 & 0 \end{pmatrix}$ Eigenvalues $\mu_{1,2} = -ip \pm \sqrt{1 - p^2}$ No damping and stable if $\lambda < 1$	$O(2)$	$-\left(1/6 - \frac{\lambda^2}{24}\right)k^2\Delta x^2 + O(k^4\Delta x^4)$

21		$\mu_1 \to 1$ Physical mode $\mu_2 \to -1$ Computational mode		
Upstream Differencing (Donor Cell)	$U_j^{n+1} = U_j^n - \lambda/2\,(U_{j+1}^n - U_{j-1}^n)$ $+ \frac{\lvert\lambda\rvert}{2}(U_{j+1}^n + U_{j-1}^n - 2U_j^n)$ Computational viscosity, $\kappa' = \frac{c\Delta x}{2}$	$G = 1 + \lvert\lambda\rvert\,(\cos(k\Delta x) - 1) - ip$ $= 1 + \lvert\lambda\rvert\left(\sqrt{1 - p^2/\lambda^2} - 1\right) - i$	$O(2)$	$-\left(\frac{1}{6} - \frac{\lvert\lambda\rvert}{2} + \frac{\lambda^2}{3}\right)k^2\Delta x^2 + O(k^2\Delta x^2)$
Lax-Wendroff	$U_j^{n+1} = U_j^n - \lambda/2\,(U_{j+1}^n - U_{j-1}^n)$ $+ \frac{\lambda^2}{2}(U_{j+1}^n + U_{j-1}^n - 2U_j^n)$ Computational viscosity, $\kappa' = \frac{c\Delta t}{2}$ (Or Two Steps)	$G = 1 - ip - \lambda^2\left(1 - \sqrt{1 - p^2/\lambda^2}\right)$ $\lvert G\rvert^2 = 1 - 4(\lambda^2 - \lambda^4)\sin^4\frac{k\Delta x}{2}$  	$O(2)$	$-\left(\frac{1}{6} - \frac{\lambda^2}{6}\right)k^2\Delta x^2 + O(k^2\Delta x^2)$
Adams-Bashforth	$U_j^{n+1} = U_j^n - 3/4\,\lambda(U_{j+1}^n - U_{j-1}^n) + \lambda/4\,(U_{j+1}^{n-1} - U_{j-1}^{n-1})$	$\mu_{1,2} = 1/2\,(1 - 3/2\,ip) \pm \sqrt{1 - 9/4\,p^2 - ip}$  	$O(2)$	
Backward (implicit)	$U_j^{n+1} = U_j^n - \lambda/2\,(U_{j+1}^{n+1} - U_{j-1}^{n+1})$	$G = \frac{1}{1+ip}$ $\lvert G\rvert^2 = \frac{1}{1+p^2}$	$O(1)$	$-\left(1/6 + \frac{\lambda^2}{3}\right)k^2\Delta x^2 + O(k^2\Delta x^2)$
Trapezoidal (Implicit)	$U_j^{n+1} = U_j^n - \lambda/2\left(\frac{U_{j+1}^n + U_{j+1}^{n+1}}{2} - \frac{U_{j-1}^n + U_{j-1}^{n+1}}{2}\right)$	$G = \frac{1 - 1/2\,ip}{1 + 1/2\,ip}$ $\lvert G\rvert^2 = 1$	$O(1)$	$-\left(1/6 + \frac{\lambda^2}{12}\right)k^2\Delta x^2 + O(k^4\Delta x^4)$

Euler – Backward (Matsuno)	2 steps . Euler – then Backward $U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\lambda^2}{4}\left(U_{j+2}^n + U_{j-2}^n - 2U_j^n\right)$ Computational viscosity $\alpha = \lambda^2 \Delta t$	$G = 1 - ip - p^2$ $\lvert G^2 \rvert = 1 - p^2 + p^4$ 	$O(1)$	$-\left(\frac{1}{6} - \frac{2\lambda^2}{3}\right) k^2\Delta x^2 + O(k^4\Delta x^4)$
Heun	2 steps Euler – Trapezoidal. $U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\lambda^2}{8}\left(U_{j+2}^n + U_{j-2}^n - 2U_j^n\right)$ Computational viscosity $\alpha = \frac{\lambda^2}{2}\Delta t$	$G = 1 - \frac{1}{2}p^2 - ip$ $\lvert G \rvert^2 = 1 + \frac{p^4}{4}$ Slightly unstable	$O(2)$	$-\left(\frac{1}{6} - \frac{\lambda^2}{6}\right) k^2\Delta x^2 + O(k^4\Delta x^4)$